A SOLUTION TO THE SCHWINGER-DYSON EQUATIONS OF QUANTUM ELECTRODYNAMICS

BY

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A non-perturbative solution to the unrenormalized Schwinger-Dyson equations of Quantum Electrodynamics was obtained by using combined analytical and numerical techniques. The photon propagator is approximated by its form near the mass shell. The vertex equation is cut off at the second order in the coupling constant and the remainder is approximated by H. S. Green's generalization of the Ward Identity for higher order diagrams. Under these approximations a functional form for the electron propagator, $S(\bar{p})$, and the vertex function, $\Gamma^{\mu}(\bar{p},\bar{p}+\bar{k})$ was obtained for all magnitudes of momenta. Both functions were found to be finite. No infinities were subtracted to obtain the solutions.

CHAPTER I

INTRODUCTION

A physical theory must furnish an accurate description of observable quantities. Beyond this supreme requirement a physical theory is judged by its simplicity and ease of comprehension. Another important measure of a theory is how widely the approach could be applied. There is a sense of a theory being more truthful if it is applicable to more than a single subject. If the same approach can be applied to several diverse problems then the problems lose their diversity. To be a really successful theory it needs to be one that unifies a broad spectrum of ideas.

Quantum Electrodynamics (Q.E.D.) has been a successful theory to the extent that it enjoys a number of these characteristics. One of its attributes is that the theory had its beginnings in the bringing together of several separate fields of study. This unification formed Relativistic Quantum theory and then as a natural extension the study of dynamics of the interaction of particles with electromagnetic radiation evolved.

Quantum Theory (Q.T.) was developed in the 1930's from the ideas of wave mechanics. It accomplished the explanation of atomic structure, molecular structure, the structure of solids and the symmetry and energy bands of crystals. However, Q.T. was insulated from the effects of special relativity in all these endeavors because the ordinary effects of chemistry arise from interactions with only the outermost part of

the atom. The calculation of reaction rates, surface potentials or scattering cross sections requires a detailed description of only the outermost electrons which require relatively low energies to excite. Relativistic corrections represent a very small fraction of these calculations.

Successful as Q.T. was at describing in terms of fundamental principles the workings of chemistry, it represented a break in the general pattern of explaining physical phenomena because it had not incorporated the principles of special relativity. Quantum Theory needed to be formulated in a Lorentz covariant form. The relativistic descendant of Schrodinger's equation is the Klein Gordon equation. This equation allows for the relativistic variation of mass with the velocity of the particle. However such effects as the fine structure of the hydrogen atom were only partly accounted for by the relativistic Q.T. of Schrodinger. It took Dirac's work on the fully relativistic theory of particles with spin 1/2 to complete the analysis of the fine structure and to explain the Zeeman effect.

A complete description of the relativistic electron would have to include the interaction of the electron with its own electromagnetic field. Feynman[1] and Schwinger[2] formed calculational methods in Q.E.D. in two separate mathematical languages. Dyson[3] demonstrated these languages were equivalent. As early as 1930, Waller, Weisskopf and Oppenheimer[4] had calculated the self energy of the electron and found it to be disappointingly, quadratically divergent. Later Weisskopf established that the divergencies were only logarithmic. These inexplicable divergencies that occurred in the calculation of measurable

quantities (though at the time such measurements were not practically feasible), held the theory in a quandary for quite a while.

Real impetus was given to Q.E.D. when Lamb and Retherford[5] succeeded in measuring the splitting between the $2S_{\frac{1}{2}}$ and the $2P_{\frac{1}{2}}$ energy levels of the hydrogen atom. Acting on a suggestion of Lamb's, Bethe[6] circumvented the divergence problem by simply cutting off the range of integration over the divergent integrals. Surprisingly, Bethe came up with a very close calculation of the "Lamb shift," as it has come to be known.

Other attempts were made at trying to eliminate the divergencies in a more rigorous manner. Schwinger and Tomonaga[29] developed the first Lorentz covariant scheme designed to make the elimination of the divergencies more acceptable. But, by whatever the justification, calculations in Q.E.D. have enjoyed remarkable practical success. Because of the small coupling constant for the electromagnetic interaction, perturbation techniques have resulted in impressive calculations of experimental values of the Lamb shift, separation of the ground state doublet of positronium, the hyperfine structure of the hydrogen atom, the line shape of emitted radiation in atomic transitions and other relativistic but measurable phenomena.

The road which connects Quantum Theory with special relativity leads to the theory of Quantum Electrodynamics. The road continues on today to connect Q.E.D. with further frontiers. Quantum Electrodynamics has become a model for other field theories. An appreciation of the special role of Q.E.D. is one way to provide for a better understanding of the mathematical structure of field theories in general. The recent success of the Non-Abelian gauge theories in unifying the

electromagnetic interaction with the weak interaction further motivates efforts to understand the ambiguities of Q.E.D. The current theories of electro-weak and strong interactions have been shown to have the same underlying structure based on assumptions about global and local invariance. The unification has its dark side since as a part of the bargain comes the problem of the divergencies

It is apparent that Q.E.D. sits as a bridge between many well-travelled roads of thought. This is why it is particularly frustrating that the theory should be flawed by unnatural infinities which rear up in the evaluation of physically observable quantities. If it were a complete and satisfactory theory they should never have occurred.

There are four different kinds of divergencies. The following classification of them has been paraphrased from the text by Janch and Rohrlich,[8]

(a) divergencies associated with the description of the vacuum

(b) infrared divergencies

(c) divergencies associated with closed loops

(d) serious divergencies.

The type (a) occurs only in the form of a phase factor multiplying a particle amplitude. Since this does alter the probability density it can have no observable effect. It is possible to ignore type (a) divergencies. Type (b) is an artifact of the mathematical procedure. An analysis of the problem has shown it can be eliminated by an improved mathematical treatment. Type (c) is associated with the photon self energy. This type of divergence has been handled by invoking the invariance of the theory under gauge transformations. Type (d), the

"Serious" divergencies, comes up in the calculation of the vacuum polarization, electron self energy and the vertex function. These serious divergencies are the object of this whole discussion.

The redressing of these infinite quantities is called Renormalization. In this process the amplitudes are expanded in a power series of the coupling constant. Many terms in the series may contain divergent integrals. It is possible to remove these infinities in a relativisticly covariant way by redefining the parameters of mass and charge. The finite terms which remain in the series are taken to be the renormalized expressions for the amplitudes. Renormalization is neither simple nor straightforward. It brings in a new and complicated set of rules which are not properly anticipated by the initial understanding of the problem. Renormalization is an after the fact reaction to something unforeseen and undesirable. This leaves two possible interpretations of the problem:

(1) There is something wrong with the basic theory of Q.E.D.

(2) Some mathematical procedure has been inappropriately applied. A new method of calculation must be tried.

It is hard to argue that there is something wrong with the foundations of the theory. Q.E.D. has enjoyed a huge practical success in calculating various physical measurements. For example, from renormalized perturbation calculations of the anomolous magnetic moment of the electron the following results for the inverse of the fine structure constant were derived:[9]

$$\frac{1}{\alpha} = 137.03549(21).$$

Presently the best experimental value is[10]

$$\frac{1}{\alpha}_{\text{exptl}} = 137.03604(11).$$

It would seem that there could be little wrong with a theory that makes such accurate predictions.

The unqualified success of Q.E.D. in calculating (by whatever means) precise experimental results, and the success Q.E.D. has enjoyed in linking quantum theory to special relativity, and its position central to general unifying theory of forces are forceful evidence that the fundamentals of Q.E.D. are sound.

The implication therefore must be that some mathematical procedure has been inappropriately applied. The response to this implication has been a long search by many persons for a self-consistent and finite approach to Q.E.D. In 1954, Gell-Mann and Low[12] sought to demonstrate that the renormalizing constants which relate the bare mass and charge are infinite. They found they could not rule out the possibility of infinite coupling constants but they isolated a necessary condition for the vacuum polarization to be finite. Johnson, Baker and Willey[13] took up the interesting problem in a long series of papers. Under a certain set of approximations they solved for the renormalizing constants and concluded that in order for the self energy of the electron to be finite, the bare mass of the electron must be zero and a special gauge must be used. These results spawned interest in a number of people[14] for finding asymptotic but non-perturbative evaluations of the self energy of the electron, the vertex function and the vacuum polarization.

Chapter III will describe the paper in which the author was involved in

some of the effort of repeating, in a new way, the determination of the electron self energy. The work reaffirmed the results of Johnson, Baker and Willey by using an inventive non-perturbative approach of H. S. Green's with less restrictive approximations than had been used before. This work extended the results of Johnson, Baker and Willey by finding a complete solution instead of an asymptotic one.

These three basic functions, the electron propagator, the vertex and the photon propagator are identified by their role in the three-linked nonlinear integral equations known as the Schwinger-Dyson equations. It is of great interest if a method of solution could be found which would yield no divergent function forms for the three basic functions. Encouraged by the success of Green's method in extracting a finite solution for the electron propagator, we decided to see if the same non-perturbative procedure of H. S. Green would work to provide a vertex function.

The description of the project is arranged in the following way. Chapter II is a general description of the overall method of taking the nonlinear linked integral equations and transforming them into a set of differential equations. An outline is given of the proposed method for solving the equations for the electron propagator and the vertex. Chapter III provides a description of the electron propagator solution. Chapter IV provides a preparation of the differential equations for the vertex function. The tensor equation is broken down into eight linked third-order differential equations for the eight scalar functions which comprise the transverse part of the vertex. In Chapter V an algebraic technique is described which makes the product of the complicated gamma

matrix function easier to obtain. In Chapter VI, Green's method is used to obtain an approximate solution to the eight linked differential equations. In Chapter VII this solution is tested and its range of applicability is defined. In Chapter VIII some alterations are made to the solution which extends its viability. The last chapter summarizes the solutions which were obtained to the Schwinger-Dyson equations.

It is felt that the work described here will contribute to the idea that Q.E.D. is a complete and satisfactory theory; a theory which is a faithful rendition of experimental results, a theory which lends clarity by unifying several fields of study, and a theory which is unambiguously expressed.

CHAPTER II

THE GENERAL PROCEDURE FOR THE SOLUTION OF THE SCHWINGER-DYSON EQUATIONS OF QUANTUM ELECTRODYNAMICS

2-1 Schwinger-Dyson Equations

Quantum Electrodynamics is a description of the interaction of light with matter. A classical charged particle generates an electromagnetic field around it with which it can self interact. Calculations of this interaction have traditionally led to infinities. To demonstrate how these infinities arise in Q.E.D., it is convenient to express the theory in terms of the propagators of the particles.

The amplitude of an electron at some point in space-time is related to its amplitude at a different point in space-time by its Feynman propagator or Green's function. The Green's function is determined by the equation of motion that governs the wave function. For the free electron the differential equation is

$$(i\not{\partial} - m)\Psi = 0 \tag{2-1}$$

where $\not{\partial}$ is a shorthand notation for $\gamma^\mu \frac{\partial}{\partial x^\mu}$ and γ^μ is a Dirac gamma matrix.[†] The photon wave equation is

$$\Box A^\mu = 0 \tag{2-2}$$

where $\Box$ is the D'ALambertian, $\frac{\partial}{\partial x_\mu} \frac{\partial}{\partial x^\mu}$. The propagator for the

[†]See Appendix A for representations of the γ^μ.

electron satisfies a corresponding equation,

$$(i\not{\partial}-m_0)S_0(x',x) = \delta^4(x'-x). \tag{2-3}$$

The solution to this equation in momentum space determines the Fourier transform of the free electron propagator to be

$$S_0(\bar{p}) = \frac{1}{\not{p}-m_0} . \tag{2-4}$$

Similarly the photon propagator satisfies

$$\Box D_0(x-x') = i\delta^4(x-x'), \tag{2-5}$$

so that the Fourier transform of the photon progagator is

$$D(q^2) = \frac{-1}{q^2} . \tag{2-6}$$

Thus the propagators for the free particles are explicitly known.

When it is allowed that a source term may be present, the interaction between the electron and photon will lead to nonhomogeneous differential equations. The exact electron and photon propagators are then determined by these nonhomogeneous differential equations but the solutions are not explicitly known. The equations which determine the Fourier transform of the photon and electron propagators are an open set of interlocked integral equations. This hierarchy of integral equations was formulated by Dyson[3] and Schwinger.[16] Using the notational practices of Bjorken and Drell,[17] these integral equations appear as:†

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†The 4 vectors are denoted by a bar over the symbol and matrices are distinguished by a bar under the symbol.

(a) $$\underline{S}(\bar{p}) = \underline{S}_0(\bar{p}) + \underline{S}_0(\bar{p})\underline{\Sigma}(\bar{p})\underline{S}(\bar{p}) \tag{2-7}$$

or equivalently

$$\underline{S}^{-1}(\bar{p}) = \underline{S}_0^{-1}(\bar{p}) - \underline{\Sigma}(\bar{p}) \tag{2-8}$$

where

$$\underline{\Sigma}(\bar{p}) = \frac{ie_0^2}{(2\pi)^4}\int \underline{\Gamma}^{\mu}(\bar{p},\bar{q})\underline{S}(\bar{q})D_{\mu\nu}(\bar{p}-\bar{q})\gamma^{\nu}d^4q. \tag{2-9}$$

(b) $$D_{\mu\nu}(k^2) = D_{o\mu\nu}(k^2) + D_{o\mu\alpha}(k^2)\Pi^{\alpha\beta}(k^2)D_{\beta\nu}(k^2) \tag{2-10}$$

where

$$\Pi_{\mu\nu}(k^2 = \frac{ie_0^2}{(2\pi)^4}\int \mathrm{Tr}\ [\gamma^{\alpha}\underline{S}(\bar{q})\underline{\Gamma}^{\beta}(\bar{q},\bar{q}+\bar{k})\underline{S}(\bar{q}+\bar{k})]\ d^4q. \tag{2-11}$$

(c) $$\underline{\Gamma}^{\mu}(\bar{p},\bar{q}) = \underline{\Gamma}_0^{\mu} + \underline{\Lambda}^{\mu}(\bar{p},\bar{q}) \tag{2-12}$$

where

$$\underline{\Lambda}^{\mu}(\bar{p},\bar{q}) = ie_0^2\int D^{\nu\eta}(k^2)\underline{\Gamma}_{\nu}(\bar{p},\bar{p}-\bar{k})\underline{S}(\bar{p}-\bar{k})$$

$$\underline{\Gamma}^{\mu}(\bar{p}-\bar{k},\bar{q}-\bar{k})\underline{S}(\bar{q}-\bar{k})\underline{\Gamma}_{\eta}(\bar{q}-\bar{k},\bar{q})\frac{d^4k}{(2\pi)^4}$$

$$+ \ldots\int\ldots\int \underline{\Gamma}^{\delta}(\bar{p},\bar{p}-\bar{k})\underline{S}(\bar{p}-\bar{k})\ \ldots\ \underline{\Gamma}^{\mu}\ \ldots$$

$$\underline{\Gamma}_{\eta}\ (\bar{q}-\bar{k}_n-\bar{k}_{n-1},\bar{q}-\bar{k}_n)\underline{S}(\bar{q}-\bar{k}_n)\underline{\Gamma}_{\nu}(\bar{q}-\bar{k},\bar{q})$$

$$d^4k_n\ \ldots\ d^4k_1(2\pi)^{-4n} + \ldots\ . \tag{2-13}$$

The zero subscript follows all bare quantities, that is, those functions or constants which are associated with the free particles. The $\underline{\Gamma}^{\mu}(\bar{p},\bar{q})$ is called the vertex function. It represents the sum of all nodeless diagrams which are connected to two external electron lines and one

external photon line. If a closed expressed for the vertex function could be formed, then a complete knowledge of the interaction propagators would depend only on a solution to the three linked nonlinear equations; the electron propagator equation, Eq. (a), the photon propagator equation, Eq. (b), and the vertex equation, Eq. (c). However, Eq. (c) gives the vertex equation in terms of an infinite series. Therefore the Schwinger-Dyson equations, though simple in form, possess no simple solution.

One method for attempting a solution is to assume the interaction propagators differ from the bare propagators by only a small variation. Perturbation theory might then lead to at least an asymptotically converging series. The first iteration would replace the propagators on the right-hand side of Eqs. (a), (b) and (c) by the bare propagators. When this is done the equation for the electron propagator function $\underline{\Sigma}(\bar{p})$ becomes

$$\underline{\Sigma}(\bar{p}) = \frac{e_o^2}{(2\pi)^4} \int \gamma^\mu \frac{1}{(\not{p}-\not{k})-m_o} \gamma^\nu \frac{g_{\mu\nu}}{k^2} d^4k$$

$$= \frac{e_o^2}{(2\pi)^4} \int \frac{4(\not{p}+\not{k}-m_o)}{[(\bar{p}+\bar{k})^2-m_o^2]} \frac{1}{k^2} d^4k. \tag{2-14}$$

By a power counting of k it can be seen that for $k\to\infty$ the integrand behaves like k^3dk/k^4 which would yield a logarithmic divergence. A more careful consideration of this integral would take into account the hyperbolic metric. The four-dimensional integral can be performed† by transforming into hyperspherical coordinates but the logarithmic divergence persists. A corresponding divergence appears in the vertex function

†See Appendix D on the subject of calculation of four-dimensional integrals.

and in the photon propagator function. These are of the type (d) category of divergences of Janch and Rohrlich[8]--the so-called "serious divergences."

These divergences can be handled by any of a number of methods grouped under the title of Renormalization methods. The first proof of renormalizability was provided by Dyson, Salam and Ward.[7] Basically the idea is that although absolute calculations cannot be made with the formalism as it stands, still relative calculations can be made. The parameters of the theory such as mass and charge are redefined to absorb the terms which contribute the infinite quantities. The renormalized mass and charge are taken to have the experimentally observed values. Renormalization, as such, was a huge but puzzling step forward. It provided the tools to make impressive calculations of relativistic corrections in the spectrum of the atom but the meaning and the value of the bare quantities remained a mystery.

The very fact that renormalization works is an indication that there ought to be a mathematically consistent way to solve the hierarchy without encountering undefinable quantities. The practical success of renormalized theory argues against a fundamental flaw in the theory. For these reasons the following scheme was developed to solve the equations without recourse to renormalization methods.

The procedure which has made possible an unrenormalized solution of the Swinger-Dyson equations has three elements to it. The first is the assumption of a reasonable starting point in terms of an approximate form for the photon propagator. The second is the generalization

of the Ward Identity which provides a neat formula for systematically truncating the hierarchy of the vertex equation at increasing levels of accuracy. The last element consists of transforming the electron and vertex integral equations into linked differential equations. The differential equations are more tractable than the integral equations to numerical and analytic approaches to the solution. A description of these three elements is the object of the following three sections.

2-2 Initial Approximations

The question which must be raised first is whether it is best to solve first for the electron propagator, $S(\bar{p})$, or first for the photon propagator, $D_{\mu\nu}(k^2)$. This can be decided by a consideration of which of the two is easiest to approximate. Since Lorentz[30] first offered the idea, as early as 1909, it has been a popular view to consider the mass of the electron as mechanical in origin. This was based on the observation that the electron, when accelerated by interaction with the electromagnetic field, behaves as though it were gaining mass. It is appealing to imagine that the bare mass of the electron may be zero and the self interaction with its own electromagnetic field is what "dresses" the electron in its apparent mass. If the bare mass is zero or very small† the self energy of the electron will be of the order of magnitude of its rest energy. In units of inverse time, the rest energy of the electron is 1.2×10^{14} megacycles. On the other hand, renormalized perturbation type calculations of the Lamb shift show the vacuum polarization contributes only about 27 megacycles. The disparity between these

†Obviously the assumption the bare mass of the electron is zero is predicated on the additional requirement that there be no other forms of interaction which contribute to the mass of the electron.

two numbers argues that the self energy of the electron may be more important than the vacuum polarization of the photon. Therefore, it is reasonable to start with an approximate form of the photon propagator and solve first equation (a) for the electron propagator.

The electron propagator is a function of the scalar $\not{p}$, $(\gamma_\mu p^\mu)$, and as such its most general form is expressible in terms of two functions, one the coefficient of unity and the other the coefficient of $\not{p}$. Because of the simplicity of Eq. (2-8) relative to Eq. (2-7), it is of interest to focus on the form of $S(\bar{p})^{-1}$,

$$S^{-1}(\bar{p}) = A(p^2) - B(p^2)\not{p}. \tag{2-15}$$

The vertex function is a matrix function of the Dirac gamma matrices and the four vectors, $\bar{p}$ and $\bar{k}$, the electron and photon momentum respectively. Its most general form is

$$\begin{aligned}\underline{\Gamma}^\lambda(\bar{p}+\bar{k},\bar{p}) = {} & \frac{p^\lambda}{p} F + \gamma^\lambda G_0 + p^\lambda \not{p} \frac{G_1}{p^2} + \frac{p^\lambda \not{k}}{pk} G_2 \\ & + i\sigma^{\alpha\beta} p_\alpha k_\beta p^\lambda \frac{H_0}{p^2 k} + i\sigma^{\lambda\alpha} p_\alpha \frac{H_1}{p} \\ & + i\sigma^{\lambda\alpha} \frac{k_\alpha}{k} H_2 + \varepsilon^{\lambda\alpha\beta\phi} \gamma^5 \gamma_\phi k_\alpha p_\beta \frac{I}{pk} \\ & + k^\lambda J + k^\lambda \frac{\not{p}}{p^2} K + k^\lambda \not{k} L + i\sigma^{\alpha\beta} p_\alpha k_\beta k^\lambda M\end{aligned} \tag{2-16}$$

where F, G_0, G_1, G_2, H_0, H_1, H_2 and I are functions of p^2, k^2 and u. The u is the angle variable defined as

$$u = \frac{p_\alpha k^\alpha}{|p^2|^{\frac{1}{2}} |k^2|^{\frac{1}{2}}} . \tag{2-17}$$

They are coefficients of that part of Γ^{λ} which is transverse to k^{λ}. Similarly J, K, L and M are functions of p^2, k^2 and u. They are coefficients of the longitudinal part of Γ^{λ}.

The general form of the photon propagator is known from its relativistic covariant properties. Since $D_{\mu\nu}(k^2)$ is a second rank tensor which depends only on the four-vector $\bar{k}$, the photon momentum, $D_{\mu\nu}(k^2)$ can be taken to have two components. One component is proportional to $k_{\mu}k_{\nu}$ and the other is proportional to $g_{\mu\nu}$. Thus

$$D_{\mu\nu}(k^2) = d_1(k^2)\, g_{\mu\nu}/k^2 + d_2(k^2)\, k_{\mu}k_{\nu}/k^4. \tag{2-18}$$

In addition to knowing the general forms of the three functions, the electron propagator, the vertex and the photon propagator, the limiting form of the functions is known on the mass shell. In the limit as the momenta approach the mass shell where $p^2 \to m^2$, $k^2 \to 0$, the functions approach the following limits,

$$\underline{S}^{-1}(\bar{p}) \to Z_2^{-1}(\not{p}-m) \tag{2-19}$$

$$\underline{\Gamma}^{\lambda}(\bar{p}+\bar{k},\bar{p}) \to Z_2^{-1}\,\gamma^{\lambda} \qquad \text{(2-20}$$

$$D_{\mu\nu}(k^2) \to Z_3\,\frac{g_{\mu\nu}}{k^2}\,. \tag{2-21}$$

The constants of proportionality to be determined by the theory are Z_2 and Z_3. First order perturbation theory finds them to be zero or infinite. They are called renormalization constants because the standard procedure is to use them to redefine the series expansions so that the divergences are absorbed.

The initial step of solving for the electron propagator requires a reasonable first estimate of the dressed function $D_{\mu\nu}$ and Γ^{λ}. The approximation that was used for the photon propagator was

$$D_{\mu\nu}(k^2) = Z_3[- \frac{g_{\mu\nu}}{k^2} - (1-b) \frac{k_\mu k_\nu}{k^4}]. \tag{2-22}$$

This choice was motivated by the results of earlier investigations.[13,14] Studies of the asymptotic forms of the propagators which were not inconsistent with finite renormalization constants found the photon propagator to be in the Landau gauge for at least the lowest order in perturbation theory. The Landau gauge is a special case (b=0) of the general form given in Eq. (2-22).

The approximation that was used for the vertex function was obtained from the Ward Identity which related the vertex to the inverse electron propagator,

$$k_\mu \underline{\Gamma}^\mu(\bar{p}+\bar{k},\bar{p}) = \underline{S}^{-1}(\bar{p}+\bar{k}) - \underline{S}^{-1}(\bar{p}). \tag{2-23}$$

In the limit as the photon momentum, $\bar{k}$, grows small this can be expressed in a differential form,

$$\underline{\Gamma}^\mu(\bar{p},\bar{p}) = \frac{\partial \underline{S}^{-1}(\bar{p})}{\partial p_\mu} . \tag{2-24}$$

As illustrated by Eq. (2-14), when the bare form of the functions is used to generate a first approximation to the dressed function, it is found that "serious" divergences occur. It is desirable then to avoid the substitution of bare quantities for dressed quantities. Therefore if the general form of the electron propagator were used in the differential Ward Identity, it would be possible to generate a general form for the vertex at vanishing $\bar{k}$,

$$\underline{\Gamma}^{\lambda}(\bar{p},\bar{p}) = \frac{\partial}{\partial p_{\lambda}}\,[A(p^2) + \not{p}B(p^2)]. \tag{2-25}$$

To solve the electron propagator equation a knowledge of the vertex function $\Gamma^{\lambda}(p+k,p)$ is needed. Equation (2-25) represents a very good approximation to the vertex function where $|k^2|$ is small. In considering the electron propagator Eqs. (2-7) and (2-9), it can be seen that this approximation is at its best when, as the argument of $D_{\mu\nu}$ vanishes, the integrand is at a maximum. This coincidence of the region of best approximation with the region of most importance argues that

$$\Gamma^{\lambda}(p+k,p) \cong \frac{\partial}{\partial p_{\lambda}}\,[A(p^2) + \not{p}B(p^2)] \tag{2-26}$$

is a reasonable first approximation for the vertex.

By using the approximations in equations (2-22) and (2-26) for the photon propagator and vertex, a solution to the electron propagator equation can be found. A description of the method of solution and its conclusions are found in the next chapter. Once a solution to the electron equation had yielded a functional form for the electron propagator functions $A(p^2)$ and $B(p^2)$, a solution to the vertex equation was sought. The next section describes the method by which the vertex equation

was approximated so that it no longer depended on the higher levels of the hierarchy.

2-3 Approximating the Vertex Equation with Green's Generalized Ward Identity

A construction of the infinite hierarchy of equations can be selected so that a highly repetitive pattern emerges which can be used to systematically separate the equations at any chosen level of complexity. Instead of the infinite series which appears in Eq. (c) the vertex equation can be written as

$$\underline{\Gamma}^{\mu} = \underline{\Gamma}^{\mu}_{o} + \underline{\Lambda}(\bar{p},\bar{q})$$

where

$$\underline{\Lambda}^{\mu}(\bar{p},\bar{q}) = \frac{ie_o^2}{(2\Pi)^4}\int \gamma^{\beta} D_{\alpha\beta}(k^2)\underline{S}(\bar{p}-\bar{k})\underline{\Gamma}^{\mu}(\bar{p}-\bar{k},\bar{p}-\bar{q}-\bar{k})$$

$$\cdot\ \underline{S}(\bar{p}-\bar{q}-\bar{k})\underline{\Gamma}^{\alpha}(\bar{p}-\bar{q}-\bar{k},\bar{p}-\bar{q})d^4k$$

$$+ \frac{ie_o^2}{(2\Pi)^4}\int \gamma^{\beta} D_{\alpha\beta}(k^2)\underline{S}(\bar{p}-\bar{k})\underline{E}^{\mu\alpha}(\bar{p}-\bar{k},\bar{q},\bar{p}-\bar{q})d^4k. \qquad (2\text{-}27)$$

A new function has appeared, $\underline{E}^{\mu\alpha}$, a function of three external momenta, which corresponds to four independent points in space-time; hence $\underline{E}^{\mu\alpha}$ is a four-point function. Just as the two-point functions $\underline{S}(\bar{p})$ and $D_{\mu\nu}(k^2)$ were related to the three-point function, $\Gamma^{\lambda}(\bar{p},\bar{p}+\bar{k})$, and as the three-point vertex, Γ^{λ}, was related to the four-point function $E^{\mu\alpha}(\bar{p}-\bar{k},\bar{q},\bar{q}-\bar{k})$, so the four-point function, $E^{\mu\alpha}$, can be related to a five-point function and so forth. The greater the number of momenta involved the lower the contribution from such a cross section should be.

Each new n-point function is created by pulling a dressed photon and dressed electron propagator out of an n-1 point diagram, creating in this way a new vertex or point.

In Table 2-1 the equations for the two-point electron propagator, the three-point vertex, and the four-point $E^{\mu\nu}$ are given to show the pattern that is emerging.

The repeated structure in the relations that link an n point diagram to an n+1 point diagram suggests there should be a generalization of the well-known Ward Identity which states

$$q_\mu \Lambda^\mu(\bar{p}-\bar{q},\bar{p}) = -\Sigma(\bar{p}-\bar{q}) + \Sigma(\bar{p}). \qquad (2\text{-}28)$$

A generalization of this for the four- and five-point diagrams would be

$$q_\mu E^{\mu\nu}(\bar{p}+\bar{q},\bar{p},\bar{k}) = \Lambda^\nu(p+q,\bar{p}-\bar{k},\bar{k}) - \Lambda^\nu(\bar{p},\bar{p}-\bar{k},\bar{k}) \qquad (2\text{-}29)$$

$$q_\mu F^{\mu\nu\alpha}(\bar{p}+\bar{q},p,k,\ell) = E^{\nu\alpha}(\bar{p}+\bar{q},\bar{k}+\bar{\ell}-\bar{p}-q,\bar{k},\bar{\ell}) - E^\nu(\bar{p},\bar{k}+\bar{\ell}-\bar{p},\bar{k},\bar{\ell}). \qquad (2\text{-}30)$$

This generalization of the Ward Identity was first proven by H. S. Green[18] in 1953. These identities exactly define the longitudinal components of the n+1 point diagram in terms of the difference of two n-point diagrams. These identities can be used to truncate the hierarchy of equations by approximating any given diagram by its longitudinal component. In this manner the vertex function could be solved for if $E^{\mu\nu}$ was approximated through an application of Eq. (2-29).

In conclusion, it has been proposed the electron propagator can be found by approximating the three-point vertex by Ward's Identity.

TABLE 2-1. RELATIONSHIP BETWEEN N-POINT DIAGRAMS

Two point diagram

$$S^{-1}(\bar{p}) = S_0^{-1}(\bar{p}) - \Sigma(\bar{p})$$

where $\underline{S}_0^{-1} = \not{p} - m_0$

$$\underline{\Sigma} = \frac{ie_0}{(2\pi)^4}\int \Gamma^{\mu}(\bar{p},\bar{p}-\bar{q})S(\bar{p}-\bar{q})D_{\mu\nu}(q^2)\gamma^{\nu}d_q^4.$$

Three point diagram

$$\underline{\Gamma}^{\mu}(\bar{p},\bar{p}-\bar{q}) = \underline{\Gamma}_0^{\mu} + \underline{\Lambda}^{\mu}(\bar{p},\bar{p}-\bar{q})$$

where $\Gamma_0 = \gamma^{\mu}$

$$\underline{\Lambda}^{\mu} = \frac{ie_0^2}{(2\pi)^4}\int \square^{\mu\alpha}(\bar{p}-\bar{k},\bar{q},\bar{p}-\bar{q})S(\bar{p}-\bar{k})D_{\alpha\nu}(k^2)\gamma^{\nu}d^4k$$

Four point diagram

$$\underline{\square}^{\mu\alpha}(\bar{p}-\bar{k},\bar{q},\bar{p}-\bar{q}) = \square_0^{\mu\alpha} + E^{\mu\alpha}(\bar{p}-\bar{k},\bar{q},\bar{p}-\bar{q})$$

where $\square_0^{\mu\alpha} = \underline{\Gamma}^{\lambda}(\bar{p}-\bar{k},\bar{p}-\bar{q}-\bar{k})\underline{S}(\bar{p}-\bar{q}-\bar{k})\Gamma^{\nu}(\bar{p}-\bar{q}-\bar{k},\bar{p}-\bar{q})$

$$E^{\mu\alpha} = \frac{ie_0^2}{(2\pi)^4}\,\underline{\pentagon}^{\mu\alpha\beta}(\bar{p}-\bar{k}-\bar{\ell},\bar{k},\bar{q},\bar{p}-\bar{q})\underline{S}(\bar{p}-\bar{\ell})D_{\mu\nu}(\ell^2)\gamma^{\nu}d^4$$

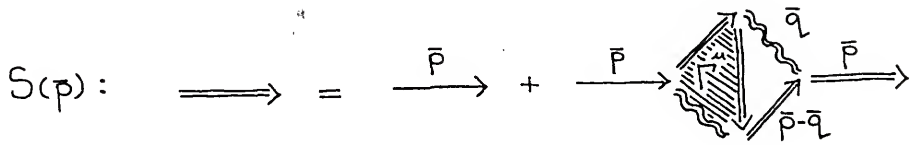
Five point diagram

$$\hexagon^{\mu\alpha\beta}(\bar{p}-\bar{k}-\bar{\ell},\bar{k},\bar{q},\bar{p}-\bar{q}) = \hexagon_0^{\mu\alpha\beta} + F^{\mu\alpha\beta}(\bar{p}-\bar{k}-\bar{\ell},\bar{k},\bar{q},\bar{p}-\bar{q})$$

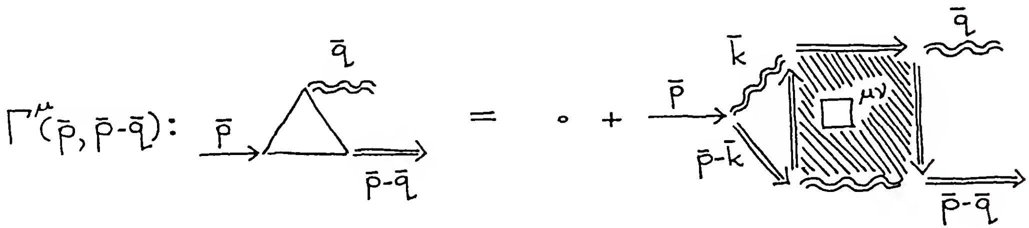
where $\hexagon_0^{\mu\alpha\beta} =$

$$F^{\mu\alpha\beta} = \frac{ie_0^2}{(2\pi)^4}\int \hexagon^{\mu\alpha\beta\phi}(\bar{p}-\bar{k}-\bar{\ell}-s,\bar{\ell},\bar{k},\bar{q},\bar{p}-\bar{q})S(p-s)D_{\mu\nu}(s^2)\gamma^{\nu}d^4s$$

Two point diagram related to the three point diagram (shaded region)



Three point diagram related to the four point diagram (shaded region)



Four point diagram related to the five point diagram (shaded region)

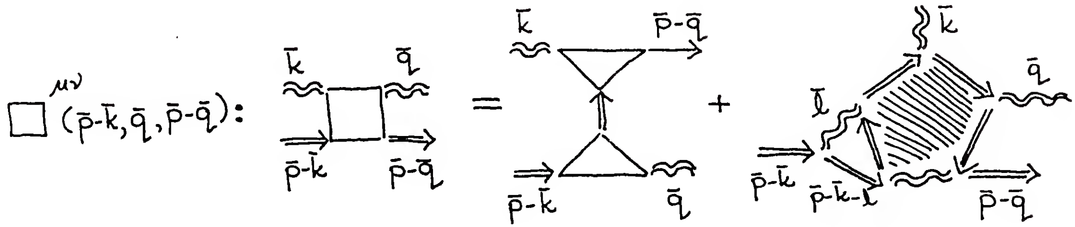


FIGURE 2-1. The Two, Three and Four Part Diagrams.

Although the Ward Identity provides an exact relationship for the longitudinal component of the vertex, it only yields the limit of the transverse part for small photon momentum. To obtain the transverse part for large values of the photon momentum a solution to the vertex equation must be found. This can be done by using the first-order generalized Ward Identity to determine an approximation for the four-point diagram. This process has two very important aspects to it. One is that at all levels in the solution to the Schwinger-Dyson equations the Ward Identity is exactly preserved. The second is never are bare functions substituted for dressed functions--a procedure which has always been associated with divergences.

2-4 Converting the Integral Equations into Differential Equations

Another essential step in the whole procedure of reducing these equations to a tractable form without giving in to the drastic approximations, which have characterized earlier attempts at a solution, was the conversion of the integral equations into differential equations. This method was developed by H. S. Green[19] in connection with the Bethe-Saltpeter equation. It was first used for a study of the Schwinger-Dyson equations for the electron by Bose and Biswas.[20]

Whenever the photon propagator appears under the integral it can be used to eliminate the integration. The photon propagator was taken to have the form,

$$D_{\mu\nu}(k^2) = -Z_3\left[\frac{g_{\mu\nu}}{k^2} + (1-b)\frac{k_\mu k_\nu}{k^4}\right]. \tag{2-31}$$

Because

$$\frac{\partial}{\partial k_\mu}\frac{\partial}{\partial k_\nu}\ln(k^2) = 2\frac{g_{\mu\nu}}{k^2} - 4\frac{k_\nu k_\mu}{k^4}\ , \tag{2-32}$$

the photon propagator can be put into the alternate form,

$$D_{\mu\nu}(k^2) = Z_3\left[-\frac{1}{4}(1-b)\frac{\partial}{\partial k_\nu}\frac{\partial}{\partial k_\mu}\ln(k^2) - \frac{1}{2}(b+1)\frac{g_{\mu\nu}}{k^2}\right]. \tag{2-33}$$

Notice also that the D'Alemberian operator,

$$\Box_k = \frac{\partial}{\partial k_\mu}\frac{\partial}{\partial k^\mu}\ , \tag{2-34}$$

will operate on $\frac{1}{k^2}$ to yield a delta function,

$$\Box_k \frac{1}{k^2} = 4\pi^2 i\delta(k^2). \tag{2-35}$$

This delta function can be used to trivially perform the integration over integrals of the form,

$$I^\lambda(k^2) = ie_0^2\int D_{\mu\nu}((\bar{q}-\bar{k})^2)\gamma^\mu F^{\lambda\nu}(q^2)\, d^4q. \tag{2-36}$$

Substituting Eq. (2-33) into Eq. (2-36) results in

$$I^\lambda(k^2) = ie_0^2\int\left[-\frac{1}{4}(1-b)\frac{\partial}{\partial(k_\nu-q_\nu)}\frac{\partial}{\partial(k_\mu-q_\mu)}\right.$$

$$\left.\ln((k-q)^2) - \frac{1}{2}(b+1)\frac{g_{\mu\nu}}{(k-q)^2}\right]\gamma^\mu F^{\lambda\nu}(q^2)d^4q. \tag{2-37}$$

Apply the D'Alembertian

$$\Box_k I^\lambda(k^2) = ie_0^2\int\left[-\frac{1}{4}(1-b)\frac{\partial}{\partial k_\nu}\frac{\partial}{\partial k_\mu}\frac{4}{(k-q)^2}\right.$$

$$\left. - \frac{1}{2}(b+1)g_{\mu\nu}4\pi^2 i\delta\ (k-q)\right]\gamma^\mu F^{\lambda\nu}(q^2)d^4q$$

$$= 2e^2(b+1)\pi^2\gamma_\nu F^{\lambda\nu}(k^2)$$

$$+ ie^2\int - (1-b)\frac{\partial}{\partial k_\nu}\not{\partial}\frac{1}{(k-q)^2}F^{\lambda\nu}(q^2)d^4q. \qquad (2\text{-}38)$$

There is one remaining part under the integral. This can be removed by either of two ways. The first is operate on the equation with $\gamma^\alpha \frac{\partial}{\partial k^\alpha} = \not{\partial}$. This yields the following third-order differential equation

$$\not{\partial}_k \Box_k I^\lambda(k^2) = 2e^2\pi^2(b+1)\not{\partial}_k\gamma_\nu F^{\lambda\nu}(k^2)$$

$$+ 4e^2\pi^2(1-b)\frac{\partial}{\partial k_\nu}F^{\lambda\nu}(k^2). \qquad (2\text{-}39)$$

The second method is to identify the remaining integral as some function $G^\lambda(k^2)$ such that

$$G^\lambda(k^2) = i\int - (1-b)\frac{\partial}{\partial k_\nu}\not{\partial}\frac{1}{(k-q)^2}F^{\lambda\nu}(q^2)d^4q. \qquad (2\text{-}40)$$

Then the problem of solving Eq. (2-38) becomes the problem of solving the pair of equations,

$$\Box_k I^\lambda(k^2) = 2e^2(b+1)\pi^2\gamma_\nu F^{\lambda\nu}(k^2) + e^2G^\lambda(k^2)$$

$$\not{\partial}G^\lambda(k^2) = 4\pi^2(1-b)\frac{\partial}{\partial k_\nu}F^{\lambda\nu}(k^2). \qquad (2\text{-}41)$$

By either of these routes a solution to an integral equation of the form (2-36) is equivalent to a solution to the differential Eq. (2-39) or the pair of differential Eqs. (2-41) when the appropriate boundary conditions are satisfied.

To briefly summarize this chapter, a scheme has been laid out by which the first two unrenormalized Schwinger-Dyson equations could be solved. The first step of the scheme involved solving for the electron propagator. To solve the electron equation required using Eq. (2-22) and (2-26) to initially approximate the vertex function and the photon function. The second stage of the scheme involved solving the vertex equation to determine the form of the transverse part of the vertex for other than very small photon momenta. To make this possible the higher order terms were also approximated using a generalization of the Ward Identity. Finally the integral equations relating these functions were to be converted into differential equations to make an analytic solution easier.

In the next chapter, Chapter III, a survey of the work that was done on the electron equation is given. In Chapter IV, an introduction to the details of solving the vertex equation is given.

CHAPTER III

THE ELECTRON PROPAGATOR EQUATION

Recall the form of the electron propagator previously given in Eq. (2-7) where S_o, the Fourier transform of the bare propagatator, is given by

$$S_o = \frac{1}{\not{p} - m_o} . \tag{3-1}$$

The m_o is the bare mass of the electron, that is, the mass the electron would have if the electromagnetic interaction could be turned off. The $\underline{\Sigma}(\bar{p})$ equation was given as

$$\underline{\Sigma}(\bar{p}) = \frac{ie_o^2}{(2\pi)^4} \int \Gamma^{\mu}(\bar{p},\bar{q}) S(\bar{q}) D_{\mu\nu}(\bar{p}-\bar{q}) \gamma^{\nu} d^4q. \tag{3-2}$$

On the basis of rationalizations detailed in section 2-2 two approximations are invoked to sever the connection of the electron equation from the vertex equation and the photon propagator equation. These were,

$$D_{\mu\nu}(\bar{p}-\bar{q}) \rightarrow \left[\frac{-g_{\mu\nu}}{(\bar{p}-\bar{q})^2} + (1-b)\frac{(p_{\mu}-q_{\mu})(p_{\nu}-q_{\nu})}{(\bar{p}-\bar{q})^4}\right] Z_3 \tag{3-3}$$

and

$$\Gamma^{\lambda}(\bar{p}+\bar{q},\bar{p}) \rightarrow \frac{\partial}{\partial p_{\lambda}} S^{-1}(\bar{p}). \tag{3-4}$$

When these substitutions are made the electron propagator Eq. (3-1) becomes

$$S^{-1}(\bar{p}) = S_o^{-1}(\bar{p}) + \frac{ie^2}{(2\pi)^4}\int \frac{\partial \underline{S}^{-1}(\bar{p})}{\partial p_\mu}\, \underline{S}(\bar{q})$$

$$\left[\frac{g_{\mu\nu}}{(\bar{p}-\bar{q})^2} - (1-b)\,\frac{(p_\mu - q_\mu)(p_\nu - q_\nu)}{(p-q)^4}\right] d^4q\gamma^\nu. \tag{3-5}$$

Because the electron propagator is a scalar function of only the electron momentum, its most general form is given by

$$S^{-1}(\bar{p}) = A(p^2) + \not{p}B(p^2). \tag{3-6}$$

So that

$$\frac{\partial}{\partial p_\mu} S^{-1}(\bar{p}) = 2p^\mu A'(p^2) + \gamma^\mu B(p^2) + 2p^\mu \not{p}\, B'(p^2) \tag{3-7}$$

where

$$A'(p^2) = \frac{dA(p^2)}{dp^2} \quad \text{and} \quad B'(p^2) = \frac{dB(p^2)}{dp^2}\ . \tag{3-8}$$

Substituting Eqs. (3-2), (3-7) and (3-8) into the electron propagator Eq. (3-6) yields,

$$A(p^2) + \not{p}B(p^2) = \not{p} - m_o + \frac{ie^2}{(2\pi)^4}\int\Big\{(2p^\mu A'(p^2)$$

$$+ \gamma^\mu B(p^2) + 2p^\mu \not{p} B'(p^2))(A(q^2) + \not{q}B(q^2))$$

$$\left[\frac{g_{\mu\nu}}{(\bar{p}-\bar{q})^2} - (1-b)\,\frac{p_\mu - q_\mu)(p_\nu - q_\nu)}{(p-q)^4}\right]\Big\}\, d^4q\gamma^\nu. \tag{3-9}$$

The equation was converted into a differential equation by application of the D'Alembertian operator, as described in section 2-4.

For the purpose of performing all of the needed matrix multiplications, a table of products of gamma matrices was prepared. This table appears in Appendix A. Equation (3-10) then separates into twolinearly independent differential equations, the coefficient equations for the unit matrix and $\not{p}$.

The unit matrix equation is

$$A''p^2 + 2A' = \frac{e^2}{4\pi^2}\left\{\frac{(4-b)}{4}\frac{AB}{(A^2-p^2B^2)} + \frac{(2+b)p^2}{4}\frac{(B'A-A'B)}{A^2-p^2B^2} - bT\right\} \qquad (3\text{-}10)$$

where

$$\tau' = -6\left\{\frac{B'A-A'B}{A^2-p^2B^2}\right\}. \qquad (3\text{-}11)$$

The $\not{p}$ equation is

$$B''p^2 + 3B' = \frac{e^2}{4\pi^2}\left\{\frac{(2+b)}{4}\frac{(A'A-B'Bp^2)}{(A^2-p^2B^2)} - \frac{(1-b)B^2}{(A^2-p^2B^2)}\right\} \qquad (3\text{-}12)$$

where

$$A' = \frac{dA}{dp^2} \quad \text{and} \quad B' = \frac{dB}{dp^2} \qquad (3\text{-}13)$$

$$A'' = \frac{d^2A}{d(p^2)^2} \quad \text{and} \quad B'' = \frac{d^2B}{d(p^2)^2}\,. \qquad (3\text{-}14)$$

Numerical solutions to these equations were formed using a fourth order Runge Kutta method which generated the values for the functions A, B and T. The functions A and B were found to be very slowly changing functions. In fact, for most practical purposes, B is essentially equal to 1. The function A very slowly declines as $p^2 \to \infty$.

A description of the solution and a simple analytic determination of the asymptotic behavior of these functions was presented in the paper[21] on the electron propagator equation. In that paper a different approach was taken. The integral equations were not converted into differential equations by the action of the D'Alembertian operator. Instead the variables of integration were converted to hyperspherical coordinates; a Wick rotation was performed; then the $(\bar{p}-\bar{q})^{-2}$ factor was expanded in terms of Gegenbaur polynomials, $C_n^{1}(\mu)$. These polynomials have an orthogonality condition which was used to simplify the integrations. This useful procedure is illustrated in detail in Appendix D.

The solutions to the differential equations had to be restricted to be particular solutions of the integral equations by the boundary conditions. The boundary conditions for the four momentum, p^2, approaching the mass shell is known. There the electron propagator is proportional to the bare propagator. The asymptotic boundary conditions, where the magnitude of p^2 is undefinitely large, are not explicitly known. This has been an object of study of a large number of papers.[14] Interesting things can be determined about the asymptotic form of the solution when the differential equations are substituted into the integral equations. This was carried out in the third section of that same paper. There it was found that the functions A and B, of the electron propagator, must approach constants for large p^2. In order that they could approach constants the gauge parameter b had to be set to zero. With b equal to zero, the photon propagator was set in the Landau gauge. At the same time it was demonstrated that a finite solution required the bare mass to be zero.

An expression for the function A was fitted to the tabulated numerical solution over a finite range of momenta. The accuracy of the fit was around 0.1 percent. To an even greater accuracy the function B was observed to equal the constant one. Thus the functions A and B appeared to be well represented by

$$A(p^2) = - |1-p^2|^{\varepsilon(1-p^2)/p^2} \tag{3-15}$$

$$B(p^2) = 1.0 \tag{3-16}$$

where $\varepsilon = 1.74517 \times 10^{-3}$.

The tabulated values of A then predicted an asymptotic form of

$$A = - |p^2|^{-\varepsilon} \tag{3-17}$$

and an asymptotic B of

$$B = 1. \tag{3-18}$$

When these asymptotic expressions were substituted back into the integral equation the power law for A was explicitly determined. It was found that

$$\varepsilon = (3\alpha/4\pi) + (3\alpha/4\pi)^2 + 3(3\alpha/4\pi)^3 + \ldots \tag{3-19}$$

where α is the fine structure constant. Baker and Johnson[13] obtained almost the same expression for the power law of A. They concluded

$$\varepsilon = (3\alpha/4\pi) + \frac{1}{6}(3\alpha/4\pi)^2 + \ldots \tag{3-20}$$

Last of all it was possible to see by comparing the limit of the electron propagator as the mass shell is approached, to the propagator expressed in terms of the renormalized propagator that the renormalization constant, Z_2, was equal to unity.

All of these resulte were in agreement with the results given by Johnson, Baker and Willey.[14] They had also concluded that the bare mass was zero and they had determined a very similar value for the power law of the asymptotic expression for A.

This paper represented an extension to the understanding of the electron propagator because it went beyond trying to determine an asymptotic form of the propagator which was consistent with finite renormalization constants. This paper actually determined an approximate expression for the electron propagator which was good for all momenta. The electron propagator was determined to be

$$S^{-1}(\bar{p}) = \not{p} + |1-p^2|^{\varepsilon(1-p^2)/p^2} \tag{3-31}$$

where $\varepsilon = 1.74517 \times 10^{-3}$.

In this manner it was shown that the electron propagator could be determined using approximations of a far less drastic nature than had been tried before. No infinite quantities were encountered. Encouraged by the success of this first stage of the project, an attempt to solve the vertex equation was ventured.

CHAPTER IV

THE VERTEX EQUATION

4-1 Introduction

The next step in the process of seeking a solution to the Schwinger-Dyson equations is solving the vertex equation, restated here,

$$\underline{\Gamma}^{\lambda}(\bar{p},\bar{q}) = \gamma^{\lambda} + \Lambda^{\lambda}(\bar{p},\bar{q}).$$

where

$$\Lambda^{\lambda}(\bar{p},\bar{q}) = \frac{ie_0^2}{(2\pi)^4}\int \gamma^{\alpha} D_{\alpha\beta}(k^2) S(\bar{p}-\bar{k})\underline{\Gamma}^{\lambda}(\bar{p}-\bar{k},\bar{q}-\bar{k})\underline{S}(\bar{q}-\bar{k})\underline{\Gamma}^{\beta}(\bar{q}-\bar{k},\bar{q})d^4k$$

$$+ \frac{ie_0^2}{(2\pi)^4}\int \gamma^{\beta} D_{\alpha\beta}(k^2)\underline{S}(\bar{p}-\bar{k})\underline{E}^{\lambda\alpha}(\bar{p}-\bar{k},\bar{q}-\bar{k},\bar{q})d^4k. \qquad (4\text{-}1)$$

As with the electron propagator equation, the solution to this equation is preceded by three elements of preparation.

The first, the photon propagator is taken to have the same form as was utilized in the electron propagator equation. However, it is now fixed in the Landau gauge so that

$$D_{\mu\nu}(k^2) = -Z_3\left[\frac{g_{\mu\nu}}{k^2} - \frac{k_{\mu}k_{\nu}}{k^4}\right]. \qquad (4\text{-}2)$$

The second element is to recognize that the four-point diagram, $E^{\beta\lambda}$, which is defined in terms of the five-point diagram, which in turn is defined in terms of all higher order diagrams, an infinite progression,

must be expressed in some closed and approximate form. Just as the vertex function could be related to the electron function through Ward's Identity, so the four-point diagram can be related to the vertex through a Generalized Ward's Identity. There are two possible longitudinal components of the four-point diagram, one is longitudinal with respect to k^{ν} and the other which is longitudinal with respect to q^{λ}. The relationships between the longitudinal components and the vertex function as given by the generalized Ward Identity are

$$k_{\nu}\underline{E}^{\lambda\nu}(\bar{p}-\bar{k},\bar{q},\bar{p}-\bar{q}) = \underline{\Lambda}^{\lambda}(\bar{p},\bar{p}-\bar{q}) - \underline{\Lambda}^{\lambda}(\bar{p}-\bar{k},\bar{p}-\bar{q}) \tag{4-3}$$

$$q_{\lambda}\underline{E}^{\lambda\nu}(\bar{p}-\bar{k},\bar{q},\bar{p}-\bar{q}) = \underline{\Lambda}^{\nu}(\bar{p}-\bar{k},\bar{p}) - \underline{\Lambda}^{\nu}(\bar{p}-\bar{k},\bar{p}-\bar{q}). \tag{4-4}$$

It is possible to substitute Γ^{λ} for Λ^{λ} and Γ^{ν} for Λ^{ν} in these relations because the difference between Γ and Λ is a constant. Using this fact and putting Eqs. (4-4) and (4-5) in the differential form it can be seen that

$$\underline{E}^{\lambda\nu}(\bar{p},\bar{q},\bar{p}-\bar{q}) = \frac{\partial}{\partial p_{\nu}}\Gamma^{\lambda}(\bar{p},\bar{p}-\bar{q}) \tag{4-5}$$

$$\underline{E}^{\lambda\nu}(\bar{p}-\bar{k},0,\bar{p}) = \frac{\partial}{\partial p_{\lambda}}\Gamma^{\nu}(\bar{p}-\bar{k},\bar{p}). \tag{4-6}$$

Thus in this procedure the complete $E^{\lambda\nu}$ will be approximated by these two parts,

$$\underline{E}^{\lambda\nu}(\bar{p}-\bar{k},\bar{q},\bar{p}-\bar{q}) \rightarrow \frac{\partial}{\partial p_{\nu}}\underline{\Gamma}^{\lambda}(\bar{p},\bar{p}-\bar{q}) + \frac{\partial}{\partial p_{\lambda}}\underline{\Gamma}^{\nu}(\bar{p}-\bar{k},\bar{p}). \tag{4-7}$$

For the purposes of simplification it is observed that the expression

$$\underline{\Gamma}^{\lambda}(\bar{p}-\bar{k},\bar{p}-\bar{q})\underline{S}(\bar{p}-\bar{q}-\bar{k})\underline{\Gamma}^{\nu}(\bar{p}-\bar{q}-\bar{k},\bar{p}-\bar{q}) - \frac{\partial}{\partial p_{\nu}}\underline{\Gamma}^{\lambda}(\bar{p},\bar{p}-\bar{q}) \tag{4-8}$$

can be reduced to

$$\frac{\partial}{\partial p_\nu} [\underline{\Gamma}^\lambda(\bar{p}-\bar{k},\bar{p}-\bar{q}-\bar{k})\underline{S}(\bar{p}-\bar{q}-\bar{k})]\ \underline{S}^{-1}(\bar{p}-\bar{q}-\bar{k}). \tag{4-9}$$

To obtain this simplification use is made of the fact that

$$\frac{\partial}{\partial p_\nu} [\underline{S}(\bar{p}-\bar{q})]\ \underline{S}^{-1}(\bar{p}-\bar{q}) = -\ \underline{S}(\bar{p}-\bar{q}) \frac{\partial}{\partial p^\nu} \underline{S}^{-1}(\bar{p}-\bar{q}), \tag{4-10}$$

and the vertex function was again approximated by

$$\underline{\Gamma}^\nu(\bar{p}-\bar{q}-\bar{k},\bar{p}-\bar{q}) \to \frac{\partial}{\partial p^\nu} \underline{S}^{-1}(\bar{p}-\bar{q}). \tag{4-11}$$

By substituting the expressions in Eqs. (4-8) and (4-9) into the vertex Eq. (4-1), one obtains a simplified vertex equation,

$$\underline{\Gamma}^\lambda(\bar{p},\bar{q}) = \gamma^\lambda + \frac{ie_o^2}{(2\pi)^4}\int \gamma^\alpha D_{\alpha\beta}(k^2)\underline{S}(\bar{p}-\bar{k}) \frac{\partial}{\partial p_\lambda} \underline{\Gamma}^\nu\ (\bar{p}-\bar{k},\bar{p})d^4k$$

$$+ \frac{ie_o^2}{(2\pi)^4}\int \gamma^\alpha D_{\alpha\beta}(k^2)\ \underline{S}(\bar{p}-\bar{k}) \frac{\partial}{\partial p_\nu} [\underline{\Gamma}^\lambda(\bar{p}-\bar{k},\bar{p}-\bar{q}-\bar{k})\underline{S}(\bar{p}-\bar{q}-\bar{k})]$$

$$\underline{S}^{-1}(\bar{p}-\bar{q}-\bar{k})d^4k. \tag{4-12}$$

The last step in preparing the vertex equation was to operate with the D'Alembertian,

$$\Box^2 = \frac{\partial}{\partial p_\nu}\frac{\partial}{\partial p^\nu}. \tag{4-13}$$

The D'Alembertian has the desirable feature of operating on the Fourier transform of photon propagator, here in the Landau gauge, to produce a Dirac delta function,

$$\square^2 D_{\mu\nu}(\bar{p}-\bar{r}) = -2\pi^2 i\, g_{\mu\nu}\ \delta(\bar{p}-\bar{r}) Z_3$$

$$- Z_3 \frac{\partial}{\partial(p^\mu - r^\mu)} \frac{\partial}{\partial(p^\nu - r^\nu)} \frac{1}{(\bar{p}-\bar{r})^2} . \tag{4-14}$$

Therefore,

$$\square^2 \underline{\Gamma}^\lambda(\bar{p},\bar{r}+\bar{p}) = \frac{ie^2}{(2\pi)^4} \int \gamma^\mu [-2\ {}^2 i\ g_{\mu\nu}\ \delta(\bar{p}-\bar{r})] \underline{S}(\bar{r})$$

$$\frac{\partial}{\partial r_\mu} [\underline{\Gamma}^\lambda(\bar{r},\bar{r}+\bar{q}) \underline{S}(\bar{r}+\bar{q})^{-1}] \underline{\ } S(\bar{r}+\bar{q}) d^4 r$$

$$+ \frac{ie^2}{(2\pi)^4} \int \gamma^\mu - \frac{\partial}{\partial(p^\mu - r^\mu)} \frac{\partial}{\partial(p^\nu - r^\nu)} \frac{1}{(p-r)^2}\ \underline{S}(\bar{r})$$

$$\frac{\partial}{\partial r_\nu} [\underline{\Gamma}^\lambda(\bar{r},\bar{r}+\bar{q}) \underline{S}(\bar{r}+\bar{q})^{-1}]\ \underline{S}(\bar{r}+\bar{q}) d^4 r$$

$$= \frac{e^2}{8\pi^2} \left\{ -\ \gamma_\nu \underline{S}(\bar{p}) \frac{\partial}{\partial p_\nu} [\underline{\Gamma}^\lambda(\bar{p},\bar{p}+\bar{q}) \underline{S}(\bar{p}+\bar{q})^{-1}]\ \underline{S}(\bar{p}+\bar{q}) \right\}$$

$$- \frac{ie^2}{(2\pi)^4} \int \gamma^\mu \left[\frac{\partial}{\partial(p^\mu - r^\mu)} \frac{\partial}{\partial(p^\nu - r^\nu)} \frac{1}{(\bar{p}-\bar{r})^2} \right] \underline{S}(\bar{r})$$

$$\frac{\partial}{\partial r_\nu} [\underline{\Gamma}^\lambda(\bar{r},\bar{r}+\bar{q}) \underline{S}(\bar{r}+\bar{q})^{-1}]\ \underline{S}(\bar{r}+\bar{q}) d^4 r. \tag{4-15}$$

In the above the definition $Z_3 e_0^2 = e^2$ was used.

At the cost of having to solve higher order differential equations the last integral can be eliminated by the action of

$$\not{\partial} = \gamma^\mu \frac{\partial}{\partial p^\mu} .$$

This yields the final form,

$$\not{\partial}^3 \Gamma^\lambda(\bar{p},\bar{r}+\bar{p}) = -\varepsilon [\tfrac{1}{2}\ \gamma_\nu\ \not{\partial}\ F^{\lambda\nu}(\bar{p},\bar{p}+\bar{q}) + \frac{\partial}{\partial p_\nu} F^{\lambda\nu}(\bar{p},\bar{p}+\bar{q})] \tag{4-16}$$

where

$$\not{\nabla}^3 = \not{\nabla}\Box, \qquad \varepsilon = e^2/4\pi^2$$

and

$$F^{\lambda\nu} = S(\bar{p}) \frac{\partial}{\partial p_\nu} [\Gamma^\lambda(\bar{p},\bar{p}+\bar{q})\, S(\bar{p}+\bar{q})^{-1}]\, S(\bar{p}+\bar{q}). \qquad (4\text{-}17)$$

4-2 <u>The Main Computer Program</u>

It is only necessary to solve the vertex equation for the transverse components of the vertex since by Ward's Identity the exact longitudinal components are known in terms of the solution to the electron equation. The main computer program is a realization of the equation

$$\not{\nabla}^3\, \underline{\Gamma}^\lambda_{trans}(\bar{p}+\bar{k},\bar{p}) = -\frac{e^2}{4\pi^2}\left[\frac{1}{2}\not{\partial}\,\gamma_\nu\, \underline{F}^{\lambda\nu}_{trans} + \frac{\partial}{\partial p^\nu}\underline{F}^{\lambda\nu}_{trans}\right]$$

where

$$\underline{F}^{\lambda\nu}_{trans} = \underline{S}(\bar{p}+\bar{k}) \frac{\partial}{\partial p_\nu} [\underline{\Gamma}^\lambda_{trans}(\bar{p}+\bar{k},\bar{p})\, \underline{S}(\bar{p})^{-1}]\, \underline{S}(\bar{p}). \qquad (4\text{-}18)$$

Hereafter the subscript "trans" will be dropped and it will be understood that any λ superscript is taken to be transverse. Thus, for any general function Q^λ,

$$Q^\lambda = Q^\lambda - k_\alpha Q^\alpha \frac{k^\lambda}{k^2}\,. \qquad (4\text{-}19)$$

In the vertex equation there appear two independent four momenta, $\bar{k}$, the photon momentum and $\bar{p}$, the electron momentum. Scalar functions therefore will be functions of the variables p^2 and

$$u = \frac{p\cdot k}{pk} \qquad (4\text{-}20)$$

where $p = |p^2|^{\frac{1}{2}}$ and $k = |k^2|^{\frac{1}{2}}$

and the functions will be parametrized by k^2. The range of p^2 is from negative to positive infinity. The variable u has the same range owing to the indefinite metric. The general expression for the vertex function in terms of such scalar functions is given by

$$\Gamma^{\lambda}(\bar{p}+\bar{k},\bar{p}) = \frac{p^{\lambda}}{p^{k}} F(p^2\mu) + p^{\lambda}[\not{p},\not{k}] \frac{H_0}{p^2k^2} (p^2,u) + [\gamma^{\lambda},\not{p}] \frac{H_1}{p^k})p^2,u)$$

$$+ [\gamma^{\lambda},\not{k}] \frac{H_2}{k^2} (p^2,u) + \gamma^{\lambda}G_0(p^2u) + p^{\lambda}\not{p} \frac{G_1}{p^2} (p^2,u)$$

$$+ p^{\lambda}\not{k} \frac{G_2}{p^k} (p^2u) + \varepsilon^{\lambda\alpha\nu\phi}\gamma^5\gamma_{\phi}p_{\nu}k_{\alpha} \frac{I}{p^k} (p^2,u). \tag{4-21}$$

In addition to the general expression for the vertex, the general expression of the Fourier transform of the electron propagator, as given in Eq. (2-16), is needed. So that

$$S(\bar{p}) = \frac{A(p^2) - \not{p}B(p^2)}{A^2(p^2) - p^2B^2(p^2)} \tag{4-22}$$

and

$$S(\bar{p}+\bar{k}) = \frac{A((p+k)^2) - (\not{p}+\not{k})B((p+k)^2)\ )}{A^2((p+k)^2) - (\not{p}+\not{k})^2B^2((p+k)^2)} . \tag{4-23}$$

For simplicity the following notation will be observed throughout,

$$A_1 = A(p^2) \qquad A_2 = A((p+k)^2)$$

$$B_1 = B(p^2) \qquad B_2 = B((p+k)^2)$$

$$D_1 = A_1^2 - p^2B^2 \qquad D_2 = A_2^2 - (p+k)^2B_2^2 . \tag{4-24}$$

Thus
$$S(\bar{p}) = \frac{A_1 - \not{p}B_1}{D_1} \tag{4-25}$$

$$S(\bar{p}+\bar{k}) = \frac{A_2 - (\not{p}+\not{k})B_2}{D_2} . \tag{4-26}$$

Equation (4-18) presents the differential vertex equation in a straightforward and simple form. Unfortunately when substitution of the scalar functions of Eqs. (4-21), (4-22) and (4-23) is made and the products of the gamma matrices are taken, thousands of terms need to be resolved. Once expanded in this way it is clear the matrix equation is equivalent to eight equations which are coefficients of the linearly independent matrices: γ^λ, $p^\lambda \not{p}$ $p^\lambda \not{k}$, $\varepsilon^{\lambda\beta\alpha\phi}\gamma^5\gamma_\phi p_\alpha k_\beta$, $i\sigma^{\alpha\lambda}k_\alpha$, $i\sigma^{\lambda\alpha}p_\alpha$, $i\sigma^{\alpha\beta}p_\alpha k_\beta p^\lambda$ and p^λ. The description of the process of identification of these eight equations will consume the next several sections. In section 4-3 the left-hand side of the eight equations will be explicitly given, and in section 4-4 the right-hand side of the eight equations will be given. These eight equations plus a set of boundary conditions will be used to determine the eight transverse vertex functions F, $G_0 G_1 G_2$, $H_0 H_1 H_2$ and I.

4-3 Left-Hand Sides of the Eight Equations

The formation of the left-hand sides of the eight equations which comprise the matrix equation for the vertex is a relatively straightforward operation. The operation of $\not{\nabla}$ on any general function, f, of p^2 and u will yield

$$\not{\nabla} f(p^2,u) = 2\not{p}f' + \left(\frac{\not{k}}{p} - \frac{u\not{p}}{p^2}\right) f^* . \tag{4-27}$$

The prime denotes a partial derivative with respect to the scalar p^2 and an asterisk denotes a partial derivative with respect to u.

When $\not{\nabla}^3$ was applied to Γ^λ the coefficients of the eight linearly independent matrices were obtained. These are given in Table 4-1.

TABLE 4-1 Left-Hand Side of the Eight Vertex Equations

Equation 1 Coefficient of p^λ

$$
\begin{aligned}
&\Big(\frac{9F}{p^4} + 23u\,\frac{F^*}{p^4} - \frac{3F^{**}}{p^4} + 10\,\frac{u^2}{p^4}\,F^{**} - u(1-u^2)\,\frac{F^{***}}{p^4} - 14\,\frac{F'}{p^2} \\
&\quad - 18\,\frac{u}{p^2}\,F^{*'} + 2(1-u^2)\,\frac{F^{**'}}{p^2} + 20\,F'' - 4u\,F^{*''} + 8\,p^2F''' \\
&- 64\,\frac{u}{p^4}\,H_0 + (30 - 86\,u^2)\,\frac{H_0^*}{p^4} + 26\,\frac{u}{p^4}(1-u^2)H_0^{***} - \frac{2}{p^4}(1-u^2)^2\,H_0^{***} \\
&\quad + 64\,\frac{u}{p^4}\,H_0' + (44\,u^2-16)\,\frac{H_0^{*'}}{p^2} - 4\,\frac{u}{p^2}(1-u^2)H_0^{**'} - 32\,u\,H_0'' \\
&\quad - 8\,(1-u^2)H_0^{*''} - 16\,p^2uH_0''' \\
&+ 18\,\frac{H_1}{p^4} + 46\,u\,\frac{H_1^*}{p^4} + (20\,u^2-6)\frac{H_1^{**}}{p^4} - 2\,\frac{u}{p^4}(1-u^2)H_1^{***} \\
&\quad - 28\,\frac{H_1'}{p^2} - 36\,\frac{u}{p^2}\,H_1^{*'} + \frac{4}{p^2}(1-u^2)H_1^{**'} + 40\,H_1'' - 8\,uH_1^{*''} \\
&\quad + 16\,p^2H_1'''\Big)\,\frac{1}{pk}. \qquad (4\text{-}28)
\end{aligned}
$$

Equation 2 Coefficient of $p^\lambda K$

$$- 8 \frac{F^*}{p^4} - 7 \frac{u}{p^4} F^{**} + 8 \frac{F^{*'}}{p^2} + 4 F^{*''} + \frac{(1-u^2)}{p^4} F^{***}$$

$$- 8 \frac{H_o}{p^4} - 4 \frac{u}{p^4} H_o^* + \frac{8}{p^2} H_o' - 28 \frac{u}{p^2} H_o^{*'} + \frac{4}{p^2} (1-u^2) H_o^{**'}$$

$$+ 64 H_o'' + 16 p^2 H_o'''$$

$$+ 8 \frac{H_1^{*'}}{p^2} - \frac{4u}{p^4} H_1^{**} - 8 \frac{H_1^*}{p^4}$$

$$+ 18 \frac{u}{p^4} H_2^* + [10 u^2 - 4(1-u^2)] \frac{H_2^{**}}{p^4} - 2u(1-u^2)\frac{H_2^{***}}{p^4} - 28 \frac{u}{p^2} H_2^{*'}$$

$$+ 48 H_2'' - 8 uH_2^{*''} + 16 p^2 H_2''' + 4(1-u^2)\frac{H_2^{**'}}{p^2} . \tag{4-29}$$

Equation 3 Coefficient of γ^λ

$$- \frac{1}{pk} [\frac{3}{p^2} F + 5 u \frac{F^*}{p^2} - (1-u^2) \frac{F^{**}}{p^2} - 8 F' + 4 p^2 F''$$

$$- \frac{8u}{p^2} H_o + 4(1-u^2)\frac{H_o^*}{p^2} + 8 uH_o'$$

$$- 12 uH_1^{*'} + 4(1-u^2)\frac{H_1^{**}}{p^2} + 20 H_1' + 4(1-u^2)H_1^{**'} + 64 p^2 H_1''$$

$$+ 16 p^4 H_1''' - \frac{8u}{p^2} H_1^*$$

$$+ (18 u^2 - 6)\frac{H_2^*}{p^2} - 14 u(1-u^2)\frac{H_2^{**}}{p^2} + 2(1-u^2)^2 \frac{H_2^{***}}{p^2}$$

$$+ (16-28u^2)H_2^{*'} + 4u(1-u^2)H_2^{**'} + 28 p^2 uH_2''$$

$$+ 16 up^4 H_2''' + 8p^2(1-u^2)H_2^{*''}]. \tag{4-30}$$

Equation 4 Coefficient of $\varepsilon^{\lambda\alpha\beta\phi}\gamma^5\gamma_\phi k_\alpha p_\beta$

$$- \frac{8}{p^4} H_o - 10 \frac{u}{p^4} H_o^{*} + \frac{2}{p^4}(1-u^2)H_o^{**} + \frac{8}{p^2} H_o^{'} + 8 H_o^{''}$$

$$- \frac{8}{p^4} H_1^{*} - 10 \frac{u}{p^4} H_1^{**} + \frac{2}{p^4}(1-u^2)H_1^{***} + \frac{8}{p^2} H_1^{*'} + 8 H_1^{*''}$$

$$- 18 \frac{u}{p^4} H_2^{*} - [10 u^2-4(1-u^2)] \frac{H_2^{**}}{p^4} + 2u(1-u^2) \frac{H_2^{***}}{p^4}$$

$$+ 28 u \frac{H_2^{*'}}{p^2} - 48 H_2^{''} + 8 uH_2^{*''} - 16 p^2 H_2^{'''} - 4(1-u^2)\frac{H_2^{**'}}{p^2}. \qquad (4\text{-}31)$$

Equation 5 Coefficient of $i\sigma^{\alpha\beta}p_\beta k_\alpha p^\lambda$

$$\frac{k}{p} [9 \frac{G_1^{*}}{p^4} + 7 \frac{u}{p^4} G_1^{**} - (1-u^2) \frac{G_1^{***}}{p^4} - 4 \frac{G_1^{*'}}{p^2} - 4 G_1^{*''}$$

$$+ 9 \frac{G_2}{g^4} + 23 u \frac{G_2^{*}}{p^4} - [3- 10 (u^2)] \frac{G_2^{**}}{p^4} - u(1-u^2) \frac{G_2^{***}}{p^4}$$

$$- 14 \frac{G_2^{'}}{p^2} - 18 \frac{u}{p^2} G_2^{*'} + 2(1-u^2) \frac{G_2^{**'}}{p^2} + 20 G_2^{''} - 4 uG_2^{*''} + 8 p^2 G_2^{'''}$$

$$+ 9 \frac{I}{p^4} + 23 u \frac{I^{*}}{p^4} - [3- 10 (u^2)] \frac{I^{**}}{p^4} - (1-u^2)u \frac{I^{***}}{p^4}$$

$$- 14 \frac{I^{'}}{p^2} - 18 u \frac{I^{*'}}{p^2} + 2(1-u^2) \frac{I^{**'}}{p^2} + 20 I^{''} - 4 uI^{*''}$$

$$+ 8 p^2 I^{'''}]. \qquad (4\text{-}32)$$

Equation 6 Coefficient of $i\sigma^{\lambda\alpha}p_\alpha$

$$9\,u\,\frac{G_o^{*}}{p^4} - (2-7u^2)\,\frac{G_o^{**}}{p^4} - u(1-u^2)\,\frac{G_o^{***}}{p^4} - 14\,u\,\frac{G_o^{*'}}{p^2}$$

$$+\,2(1-u^2)\,\frac{G_o^{**'}}{p^2} + 24\,G_o'' - 4\,u\,G_o^{*''} + 8\,p^2G_o'''$$

$$+\,4\,\frac{G_1}{p^4} + 5\,u\,\frac{G_1^{*}}{p^4} - (1-u^2)\,\frac{G_1^{**}}{p^4} - 4\,\frac{G_1'}{p^2} - 4\,G_1''$$

$$-\,9\,u\,\frac{I}{p^4} - [23\,(u^2) \;-\; 8]\,\frac{I^{*}}{p^4} + 10(1-u^2)u\,\frac{I^{**}}{p^4}$$

$$-\,(1-u^2)^2\,\frac{I^{***}}{p^4} + 14\,u\,\frac{I'}{p^2} - (8-18u^2)\,\frac{I^{*'}}{p^2}$$

$$-\,2u(1-u^2)\,\frac{I^{**'}}{p^2} - 20\,uI'' - 4(1-u^2)I^{*''} - 8\,p^2uI''' . \qquad (4\text{-}33)$$

Equation 7 Coefficient of $i\sigma^{\lambda}k_\alpha$

$$\frac{k}{p}\,[3\,\frac{G_o^{*}}{p^2} + 5\,u\,\frac{G_o^{**}}{p^2} - (1-u^2)\,\frac{G_o^{***}}{p^2} - 8\,G_o^{*'} - 4\,p^2G_o^{*''}$$

$$-\,3\,\frac{G_2}{p^2} - 5\,u\,\frac{G_2^{*}}{p^2} + (1-u^2)\,\frac{G_2^{**}}{p^2} + 8\,G_2'$$

$$+\,4\,p^2G_2'' - 3\,\frac{I}{p^2} - 5\,u\,\frac{I^{*}}{p^2} + (1-u^2)\,\frac{I^{**}}{p^2}$$

$$-\,2\,I' + 10\,u\,I^{*'} - 2(1-u^2)\,I^{**'} - 28\,p^2I'' - 8\,p^4I''']. \qquad (4\text{-}34)$$

Equation 8 Coefficient of p^{λ}

$$9\,u\,\frac{\overset{*}{G_o}}{p^4} - [2 - 7\,(u^2)]\,\frac{\overset{**}{G_o}}{p^4} - u(1-u^2)\,\frac{\overset{***}{G_o}}{p^4} - 14\,u\,\frac{\overset{*\prime}{G_o}}{p^2}$$

$$+ 2(1-u^2)\,\frac{\overset{**\prime}{G_o}}{p^2} + 24\,G_o'' - 4\,u\,\overset{*''}{G_o} + 8\,p^2 Go'''$$

$$- 12\,\frac{G_1}{p^4} - 15\,u\,\frac{\overset{*}{G_1}}{p^4} + 3(1-u^2)\,\frac{\overset{**}{G_1}}{p^4} + 12\,\frac{G_1'}{p^2} - 10\,u\,\frac{\overset{*\prime}{G_1}}{p^2}$$

$$+ 2(1-u^2)\,\frac{\overset{**\prime}{G_1}}{p^2} + 36\,G_1'' + 8\,p^2\,G_1''' + 9\,u\,\frac{G_2}{p^4}$$

$$- [8 - 23\,u^2]\,\frac{\overset{*}{G_2}}{p^4} - 10\,u(1-u^2)\,\frac{\overset{**}{G_2}}{p^4} + (1-u^2)^2\,\frac{\overset{***}{G_2}}{p^4}$$

$$- 14\,u\,\frac{G_2'}{p^2} + (8-18\,u^2)\,\frac{\overset{*\prime}{G_2}}{p^2} + 2\,u(1-u^2)\,\frac{\overset{**\prime}{G_2}}{p^2}$$

$$+ 20\,u\,G_2'' + 4(1-u^2)\overset{*''}{G_2} + 8\,p^2 u\,G_2'''. \qquad (4\text{-}35)$$

4-4 The Right-Hand Sides of the Eight Equations

The right-hand side of the vertex equation,

$$- \varepsilon[\tfrac{1}{2}\,\not{k}\,\gamma_{\nu}F^{\lambda\nu} + \frac{\partial}{\partial p^{\nu}}\,F^{\lambda\nu}], \qquad (4\text{-}36)$$

is compiled through multiple layers of matrix multiplication. It is best represented, not by an exhaustive itemization of each and every term but by definition of the various layers.

The first layer is to define the matrix function F

$$F^{\lambda\nu} = S(\bar{p}+\bar{k})\,\frac{\partial\Gamma^{\lambda}}{\partial p_{\nu}}\,(\bar{p}+\bar{k},\bar{p}) + S(\bar{p}+\bar{k})\Gamma^{\lambda}(\bar{p}+\bar{k},\bar{p})\,\frac{\partial}{\partial p_{\nu}}\,S(\bar{p})S^{-1}(\bar{p})$$

$$= S(\bar{p}+\bar{k})\left\{\frac{\partial\Gamma^\lambda}{\partial p_\nu} + \Gamma^\lambda\left[p^\nu\left(-2\frac{A_2A_2'}{D_2} + 2p^2\frac{B_2'B_2}{D_2} + \frac{B_2^2}{D_2}\right)\right.\right.$$

$$\left.\left. - \gamma^\nu\frac{A_2B_2}{D_2} + i\sigma^{\nu\alpha}p_\alpha\frac{B_2^2}{D_2} + 2p^\nu\not{p}\left(\frac{A_2'B_2}{D_2} - \frac{B_2'A_2}{D_2}\right)\right]\right\}$$

$$= S(\bar{p}+\bar{k})Z^{\lambda\nu} \tag{4-37}$$

where

$$Z^{\lambda\nu} = \frac{\partial\Gamma^\lambda}{\partial p_\nu} + \Gamma^\lambda[-2p^\nu PPD + 2p^\nu\not{p}MPD - \gamma^\nu ABD + \not{p}\gamma^\nu BBD] \tag{4-38}$$

where

$$PPD + \frac{A_2A_2'}{D_2} - p^2\frac{B_2B_2'}{D_2} \tag{4-39}$$

$$MPD = \frac{A_2'B_2}{D_2} - \frac{B_2'A_2}{D_2} \tag{4-40}$$

$$ABD = \frac{A_2B_2}{D_2} \tag{4-41}$$

$$BBD = \frac{B_2^2}{D_2}\ . \tag{4-42}$$

The $Z^{\lambda\nu}$ is a tensor with twenty-eight different linearly independent combinations of the available matrices. These twenty-eight form a group, fourteen elements of which are odd in gamma matrices, fourteen elements are even.

$$Z^{\lambda\nu}_{odd} = \not{p}g^{\lambda\nu}Z_1 + \not{p}p^\lambda p^\nu Z_2 + \not{p}p^\lambda k^\nu Z_3 + g^{\lambda\nu}\not{k}Z_4$$

$$+ \not{k}p^\lambda p^\nu Z_5 + \not{k}p^\lambda k^\nu Z_6 + p^\lambda\gamma^\nu Z_7 + \gamma^\lambda p^\nu Z_8$$

$$+ \gamma^{\lambda} k^{\nu} Z_9 + \varepsilon^{\nu\alpha\beta\phi} \gamma^5 \gamma_{\phi} k_{\alpha} p_{\beta} p^{\lambda} Z_{10} + \varepsilon^{\lambda\alpha\beta\phi} \gamma^5 \gamma_{\phi} k_{\alpha} p_{\beta} p^{\nu} Z_{11}$$

$$+ \varepsilon^{\lambda\nu\beta\phi} \gamma^5 \gamma_{\phi} p_{\beta} Z_{12} + \varepsilon^{\lambda\alpha\nu\phi} \gamma^5 \gamma_{\phi} k_{\alpha} Z_{12}$$

$$+ \varepsilon^{\lambda\alpha\beta\phi} \gamma^5 \gamma_{\phi} p_{\beta} k_{\alpha} k^{\nu} Z_{14}. \tag{4-43}$$

$$Z^{\lambda\nu}_{even} = p^{\lambda} p^{\nu} Z_{e1} + p^{\lambda} k^{\nu} Z_{e2} + \varepsilon^{\lambda\alpha\nu\beta} \gamma^5 k_{\alpha} p_{\beta} Z_{e3}$$

$$+ i\sigma^{\nu\lambda} Z_{e4} + g^{\lambda\nu} Z_{e5} + i\sigma^{\alpha\beta} k_{\alpha} p_{\beta} p^{\lambda} p^{\nu} Z_{e6}$$

$$+ p^{\lambda} k^{\nu} i\sigma^{\alpha\beta} k_{\alpha} p_{\beta} Z_{e7} + i\sigma^{\lambda\alpha} p_{\alpha} p^{\nu} Z_{e8}$$

$$+ i\sigma^{\alpha\lambda} k_{\alpha} k^{\nu} Z_{e9} + i\sigma^{\alpha\nu} k_{\alpha} p^{\lambda} Z_{e10} + i p^{\lambda} \sigma^{\nu\alpha} p_{\alpha} Z_{e11}$$

$$+ i\sigma^{\alpha\lambda} p^{\lambda} k_{\alpha} Z_{e12} + i\sigma^{\lambda\alpha} p_{\alpha} k^{\lambda} Z_{e13} + i\sigma^{\alpha\beta} k_{\alpha} p_{\beta} g^{\nu\lambda} Z_{e14} \tag{4-44}$$

where

$$Z_1 = - G_o BBD + \frac{G_1}{p^2} + 2 \frac{H_1}{p^k} ABD - uI\ BBD \tag{4-45}$$

$$Z_2 = 2F \frac{MPD}{pk} - 2 \frac{\dot{G_1}}{p^4} + 2 \frac{G_1'}{p^2} - u \frac{G_1^*}{p^4} - 2 \frac{G_1\ PPD}{p^2}$$

$$+ 4u \frac{H_\phi}{p^k} MPD - 4 \frac{H_1}{p^k} MPD \tag{4-46}$$

$$Z_3 = \frac{k\dot{G}_1^*}{p^3} - \frac{k}{p} G_2\ BBD - 2 \frac{H_o}{p^2} ABD + \frac{k}{p} I\ BBD \tag{4-47}$$

$$Z_4 = \frac{k}{p} G_2 + 2 H_2 ABD + pk\ I\ BBD \tag{4-48}$$

$$Z_5 = - \frac{k}{p^3} G_2 - u \frac{k}{p^3} G_2^* + \frac{k}{p} G_2\ BBD + 2 \frac{k}{p} G_2' - 2 \frac{k}{p} G_W\ PPD$$

$$+ 2 \frac{H_o}{p^2} ABD - 4 H_o\ MPD - 4 H_2\ MPD - \frac{k}{p} I\ BBD \tag{4-49}$$

$$Z_6 = \frac{G_2^*}{p^2} \tag{4-50}$$

$$Z_7 = - \frac{F}{pk} \, ABD + G_o \; BBD + \frac{G_1}{p^2} + G_1 \; BBD + uG_2 \; BBD \tag{4-51}$$

$$Z_8 = 2 \; G_o' - \frac{u}{p^2} \, G_o^* - 2 \; G_o \; PPD + G_o \; BBD + 4 \, \frac{p}{k} \, H_1 \; MPD$$

$$- 2 \, \frac{H_1}{pk} \, ABD + 4 \, \frac{p}{k} \, uH_2 \; MPD \; + uI \; BBD \tag{4-52}$$

$$Z_9 = k \, \frac{G_o^*}{p} - 2 \; H_2 \; ABD - p \, k \; I \; BBD \tag{4-53}$$

$$Z_{10} = - \; \frac{k}{p} \, G_2 \; BBD - 2 \, \frac{H_o}{p^2} \, ABD + \frac{k}{p} \; I \; BBD \tag{4-54}$$

$$Z_{11} = - \; 4 \; H_2 \; MPD - \frac{k}{p^3} \; I + 2 \, \frac{k}{p} \; I' - u \, \frac{k}{p^3} \; I^* - 2 \frac{k}{p} \; I \; PPD \tag{4-55}$$

$$Z_{12} = G_o \; BBD - 2 \, \frac{H_1}{p^k} \, ABD + u \; I \; BBD \tag{4-56}$$

$$Z_{13} = 2 \; H_2 \; ABD + \frac{k}{p} \; I + p \, k \; I \; BBD \tag{4-57}$$

$$Z_{14} = \frac{I^*}{p^2} \tag{4-58}$$

$$Z_{e1} = - \frac{F}{kp^3} + 2 \, \frac{F'}{pk} - u \, \frac{F^*}{kp^3} - 2 \, \frac{F}{pk} PPD + \frac{F}{pk} \; BBD$$

$$+ 2 \; G_o \; MPD + 2 \; G_1 \; MPD - \frac{G_1}{p^2} \, ABD + 2 \; uG_2 \; MPD$$

$$+ 2 \; u \, \frac{H_o}{pk} \; BBD - 2 \, \frac{H_1}{pk} \; BBD + 2 \; uG_2 \; MPD + 2 \, u \, \frac{H_o}{pk} \; BBD$$

$$+ 2 \, \frac{H_1}{pk} \; BBD \tag{4-59}$$

$$Ze_2 = \frac{F^*}{p^2} - \frac{k}{p} G_2 \text{ ABD} - 2 H_o \text{ BBD} - 2 H_2 \text{ BBD} \tag{4-60}$$

$$Z_{e3} = \frac{k}{p} I \text{ ABD} + 2 H_2 \text{ BBD} \tag{4-61}$$

$$Z_{e4} = - G_o \text{ ABD} + 2 \frac{H_1}{pk} + 2 \frac{p}{k} H_1 \text{ BBD} + 2 \frac{p}{k} u H_2 \text{ BBD} \tag{4-62}$$

$$Z_{e5} = \frac{F}{pk} - G_o \text{ ABD} + 2 \frac{p}{k} H_1 \text{ BBD} + 2 \frac{p}{k} uH_2 \text{ BBD} \tag{4-63}$$

$$Z_{e6} = - 2 \frac{k}{p} G_2 \text{ MPD} - 4 \frac{H_o}{p^4} + 4 \frac{H_o'}{p^2} - 2 u \frac{H_o^*}{p^4}$$

$$- 4 \frac{H_o}{p^2} \text{ PPD} + 2 \frac{k}{p} I \text{ MPD} \tag{4-64}$$

$$Z_{e7} = 2 \frac{H_o^*}{p^3 k} \tag{4-65}$$

$$Z_{e8} = - 2 G_o \text{ MPD} + 2 \frac{H_1}{p^3 k} - 4 \frac{H_1'}{pk} + 2 u \frac{H_1^*}{p^3 k} + 4 \frac{H_1}{pk} \text{ PPD}$$

$$- 2 u I \text{ MPD} \tag{4-66}$$

$$Z_{e9} = 2 \frac{H_2^*}{pk} \tag{4-67}$$

$$Z_{e10} = \frac{k}{p} G_2 \text{ ABD} + 2 \frac{H_o}{p^2} + 2 H_o \text{ BBD} + 2 H_2 \text{ BBD} \tag{4-68}$$

$$Z_{e11} = \frac{F}{pk} \text{ BBD} - \frac{G_1}{p^2} \text{ ABD} + 2 \frac{u}{pk} H_o \text{ BBD} - 2 \frac{H_1}{pk} \text{ BBD} \tag{4-69}$$

$$Z_{e12} = 4 H_2' - 2 \frac{u}{p^2} H_2^* - 4 H_2 \text{ PPD} + 2 H_2 \text{ BBD} - 2 pk I \text{ MPD}$$

$$+ \frac{k}{p} I \text{ ABD} \tag{4-70}$$

$$Z_{e13} = -2\frac{H_1^*}{p^2} + 2\,H_2\,BBD + \frac{k}{p}\,I\,ABD \tag{4-71}$$

$$Z_{e14} = 2\frac{H_o}{p^2} - 2\,H_2\,BBD - \frac{k}{p}\,I\,ABD \tag{4-72}$$

Since

$$F^{\lambda\nu} = S(p^2)Z^{\lambda\nu}$$

$$= \frac{A_1}{D_1}Z^{\lambda\nu} - \frac{B_1}{D_1}(\not{p}+\not{k})Z^{\lambda\nu} \tag{4-73}$$

$$\begin{aligned} F^{\lambda\nu}_{even} = {} & p^\lambda p^\nu W_1 + p^\lambda k^\nu W_2 + \varepsilon^{\lambda\alpha\nu\beta}\gamma^5 k_\alpha p_\beta W_3 + i\sigma^{\nu\lambda}W_4 \\ & + g^{\lambda\nu}W_5 + p^\lambda p^\nu i\sigma^{\alpha\beta}k_\alpha p_\beta W_6 + p^\lambda p_\alpha i\sigma^{\alpha\beta}k_\alpha p_\beta k^\nu W_7 \\ & + i\sigma^{\lambda\alpha}p_\alpha p^\nu W_8 + i\sigma^{\alpha\lambda}k_\alpha k^\nu W_9 + i\sigma^{\alpha\nu}p^\lambda k_\alpha W_{10} \\ & + ip^\lambda\sigma^{\nu\alpha}p_\alpha W_{11} + i\sigma^{\alpha\lambda}k_\alpha p^\nu W_{12} + i\sigma^{\lambda\alpha}p_\alpha k^\nu W_{13} \\ & + i\sigma^{\alpha\beta}k_\alpha p_\beta g^{\nu\lambda}W_{14} \end{aligned} \tag{4-74}$$

$$\begin{aligned} F^{\lambda\nu}_{odd} = {} & \not{p}g^{\lambda\nu}V_1 + \not{p}p^\lambda p^\nu V_2 + \not{p}p^\lambda k^\nu V_3 + g^{\lambda\nu}\not{k}V_4 \\ & + \not{k}p^\lambda p^\nu V_5 + p^\lambda \not{k}k^\nu V_6 + p^\lambda\gamma^\nu V_7 + \gamma^\lambda p^\nu V_8 \\ & + \gamma^\lambda k^\nu V_9 + \varepsilon^{\nu\alpha\beta\phi}\gamma^5\gamma_\phi k_\alpha p_\beta p^\lambda V_{10} \\ & + \varepsilon^{\lambda\alpha\beta\phi}\gamma^5\gamma_\phi p_\beta k_\alpha p^\nu V_{11} + \varepsilon^{\lambda\nu\beta\phi}\gamma^5\gamma_\phi p_\beta V_{12} \\ & + \varepsilon^{\lambda\alpha\nu\phi}\gamma^5\gamma_\phi k_\alpha V_{13} + \varepsilon^{\lambda\alpha\beta\phi}\gamma^5\gamma_\phi p_\beta k_\alpha k^\nu V_{14} \end{aligned} \tag{4-75}$$

where

$$W_1 = AD\,Z_{e1} + BD[-(p^2+pk\,u)Z_2 - (1+\frac{pu}{k})Z_5 - Z_7 - Z_8] \tag{4-76}$$

$$W_2 = AD\,Z_{e2} + BD[-(p^2+pk\,u)Z_3 - (k^2+pk\,u)Z_6 - k^2\,Z_7 - Z_9 \tag{4-77}$$

$$W_3 = AD\,Z_{e3} + BD[Z_{13} + k^2 Z_{12}] \tag{4-78}$$

$$W_4 = AD\ Z_{e4} + DB[(p^2+pku)Z_{12} - (1 + \frac{p}{k}\ u)Z_{13} \tag{4-79}$$

$$W_5 = AD\ Z_{e5} + BD[-\ (p^2+pku)Z_1 - (1 + \frac{p}{k}\ u)Z_4] \tag{4-80}$$

$$W_6 = AD\ Z_{e6} + BD[k^2Z_2 - Z_5 - Z_{10} - Z_{11}] \tag{4-81}$$

$$W_7 = AD\ Z_{e7} + BD[Z_e - Z_6 - Z_{10} - Z_{14}] \tag{4-82}$$

$$W_8 = AD\ Z_{e8} + BD[-\ Z_8 + (1 + \frac{p}{k}\ u)Z_{11} + Z_{12}] \tag{4-83}$$

$$W_9 = AD\ Z_{e9} + BD[Z_9 + (p^2+pku)Z_{14} + Z_{13}] \tag{4-84}$$

$$W_{10} = AD\ Z_{e10} + BD[k^2Z_7 + (p^2+pku)Z_{10} - Z_{13}] \tag{4-85}$$

$$W_{11} = AD\ Z_{e11} + BD[-\ Z_7 + (1 + \frac{p}{k}\ u)Z_{10} - Z_{12}] \tag{4-86}$$

$$W_{12} = AD\ Z_{e12} + BD[k^2Z_8 + (p^2+pku)Z_{11} + Z_{13}] \tag{4-87}$$

$$W_{13} = AD\ Z_{e13} + BD[-\ Z_9 + (k^2+pku)Z_{14} + k^2Z_{12}] \tag{4-88}$$

$$W_{14} = AD\ Z_{e14} + BD[k^2Z_1 - Z_4] \tag{4-89}$$

and where

$$V_1 = AD\ Z_1 + BD[-\ Z_{e5} + \frac{p}{k}\ u\ Z_{e14} + Z_{e14}] \tag{4-90}$$

$$V_2 = AD\ Z_2 + BD[-\ Z_{e1} + (\frac{p}{k}\ u+1)Z_{e6} + Z_{e8} + Z_{e11}] \tag{4-91}$$

$$V_3 = AD\ Z_3 + BD[\ -\ Z_{e_2} + pku\ Z_{e7} + Z_{e13} + k^2Z_{e7} + k^2Z_{e11}] \tag{4-92}$$

$$V_4 = AD\ Z_4 + BD[-\ p^2Z_{e14} - k^2Z_{e5} - pku\ Z_{e14}] \tag{4-93}$$

$$V_5 = AD\ Z_5 + BD[-\ p^2Z_{e6} - Z_{e10} - Z_{e12} - k^2Z_{e1} - pku\ Z_{e6}] \tag{4-94}$$

$$V_6 = AD\ Z_6 + BD[-\ (p^2+pku)Z_{e7} - Z_{e9} - Z_{e2} - Z_{e10}] \tag{4-95}$$

$$V_7 = AD\ Z_7 + BD[-\ Z_{e4} + \frac{p}{k}\ u\ Z_{e10} - (p^2+pku)Z_{e11} + Z_{e10}] \tag{4-96}$$

$$V_8 = AD\ Z_8 + BD[Z_{e4} - (p^2-pku)Z_{e8} + (1 + \frac{p}{k}\ u)Z_{e12}] \tag{4-97}$$

$$V_9 = AD\ Z_9 + BD[(k^2+pku)Z_{e9} - (p^2+pku)Z_{e13} + k^2 Z_{e4}] \tag{4-98}$$

$$V_{10} = AD\ Z_{10} + BD[Z_{e3} + Z_{e10} + k^2 Z_{e11}] \tag{4-99}$$

$$V_{11} = AD\ Z_{11} + BD[Z_{e12} - Z_{e3} + k^2 Z_{e8}] \tag{4-100}$$

$$V_{12} = AD\ Z_{12} + BD[(1 + \frac{p}{k}\ u)Z_{e3} + Z_{e4}] \tag{4-101}$$

$$V_{13} = AD\ Z_{13} + BD[-\ k^2 Z_{e4} + (p^2 pku)Z_{e3}] \tag{4-102}$$

$$V_{14} = AD\ Z_{14} + BD[Z_{e9} + Z_{e13} - Z_{e3}] \tag{4-103}$$

Another way to write the right-hand side of the vertex equation is to notice

$$-\ \varepsilon[\frac{1}{2}\ \not{k}\ \gamma_\nu F^{\lambda\nu} + \frac{\partial F^{\lambda\nu}}{\partial p_\nu}\ ]$$

$$= -\ \varepsilon\ \frac{\partial}{\partial p_\alpha}\ \{[\frac{3}{2}\ g_{\alpha\nu} - \frac{i\sigma}{2}\ \alpha\nu]F^{\lambda\nu}\}. \tag{4-104}$$

The next layer of definition is to choose

$$T^{\lambda\alpha} = [\frac{3}{2}\ g_{\ \alpha\nu} - \frac{i}{2}\ \sigma_{\alpha\nu}]F^{\lambda\nu} \tag{4-105}$$

where

$$T_{even} = p^\lambda p^\nu T_{e_1} + p^\lambda k^\nu T_{e_2} + \varepsilon^{\lambda\alpha\nu\beta}\gamma^5 k_\alpha p_\beta T_{e3}$$

$$+\ i\sigma^{\nu\lambda}T\ e4 + g^{\lambda\nu}T_{e_5} + p^\lambda p^\nu T_{e_6} + i\sigma^{\alpha\beta}k_\beta p^\lambda p^\nu k\ T_{e7}$$

$$+\ i\sigma^{\lambda\alpha}p_\alpha p^\nu T_{e8} + i\sigma^{\alpha\lambda}k_\alpha k^\nu T_{e9} + i\sigma^{\alpha\nu}p^\lambda T_{e10}$$

$$+ i\sigma^{\nu\alpha}p_\alpha p^\lambda T_{e11} + i\sigma^{\alpha\lambda}p^\nu T_{e12} + i\sigma^{\lambda\alpha}p_\alpha k^\nu T_{e13}$$

$$+ i\sigma^{\alpha\beta}k_\alpha p_\beta g^{\lambda\nu}T_{e14} \tag{4-106}$$

$$T^{\lambda\alpha}_{odd} = \not{p}g^{\lambda\nu}T_{o1} + \not{p}p^\lambda p^\nu T_{o2} + \not{p}p^\lambda k^\nu T_{o3} + g^{\lambda\nu}\not{k}T_{o4}$$

$$+ \not{k}p^\lambda p^\nu T_{o5} + \not{k}p^\lambda k^\nu T_{o6} + p^\lambda\gamma^\nu T_{o7} + \gamma^\lambda p^\nu T_{o8}$$

$$+ \gamma^\lambda k^\nu T_{o9} + \varepsilon^{\nu\alpha\beta\phi}\gamma^5\gamma_\phi k_\alpha p_\beta p^\lambda T_{o10}$$

$$+ \varepsilon^{\lambda\alpha\beta\phi}\gamma^5\gamma_\phi k_\alpha p_\beta p^\nu T_{o11} + \varepsilon^{\lambda\nu\beta\phi}\gamma^5\gamma_\phi p_\beta T_{o12}$$

$$+ \varepsilon^{\lambda\alpha\nu\phi}\gamma\ \gamma_\phi k_\alpha T_{o13} + \varepsilon^{\gamma\alpha\beta\phi}\gamma^5\gamma_\phi k_\alpha p_\beta T_{o14} \tag{4-107}$$

where the

$$T_{e1} = \frac{3}{2}W_1 - \frac{1}{2}\frac{pu}{k}W_6 - \frac{1}{2}W_7 - \frac{1}{2}W_8 - \frac{3}{2}W_{11} \tag{4-108}$$

$$T_{e2} = \frac{3}{2}W_2 + \frac{1}{2}p^2 W_6 + \frac{1}{2}pku\, W_7 + \frac{3}{2}W_{10} + \frac{1}{2}W_{12} + \frac{1}{2}W_{14} \tag{4-109}$$

$$T_{e3} = \frac{3}{2}W_3 - \frac{1}{2}W_{12} - \frac{1}{2}W_{13} + \frac{1}{2}W_{14} \tag{4-110}$$

$$T_{e4} = \frac{5}{2}W_4 - \frac{1}{2}W_5 + \frac{1}{2}W_9 + \frac{1}{2}\frac{pu}{k}W_{12} - \frac{1}{2}p^2W_8 - \frac{1}{2}\frac{pu}{k}W_{13} \tag{4-111}$$

$$T_{e5} = \frac{3}{2}W_5 - \frac{3}{2}W_4 - \frac{1}{2}\frac{pu}{k}W_{12} + \frac{1}{2}p^2W_8 + \frac{1}{2}\frac{pu}{k}W_{13} - \frac{1}{2}W_9 \tag{4-112}$$

$$T_{e6} = W_6 \tag{4-113}$$

$$T_{e7} = W_7 \tag{4-114}$$

$$T_{e8} = W_8 \tag{4-115}$$

$$T_{e9} = W_9 \tag{4-116}$$

$$T_{e10} = \frac{5}{2} W_{10} + \frac{1}{2} W_2 + \frac{1}{2} p^2 W_6 + \frac{1}{2} pku\, W_7 + \frac{1}{2} W_{14} + \frac{1}{2} W_{12} \qquad (4\text{-}117)$$

$$T_{e11} = \frac{5}{2} W_{11} - \frac{1}{2} W_1 + \frac{1}{2} \frac{pu}{k} W_6 + \frac{1}{2} W_7 + \frac{1}{2} W_3 \qquad (4\text{-}118)$$

$$T_{e12} = \frac{3}{2} W_{12} - \frac{1}{2} W_3 - \frac{1}{2} W_{14} + \frac{1}{2} W_{13} \qquad (4\text{-}119)$$

$$T_{e13} = \frac{3}{2} W_{13} - \frac{1}{2} W_3 - \frac{1}{2} W_{14} + \frac{1}{2} W_{12} \qquad (4\text{-}120)$$

$$T_{e14} = \frac{3}{2} W_{14} + \frac{1}{2} W_3 - \frac{1}{2} W_{12} - \frac{1}{2} W_{13} \qquad (4\text{-}121)$$

$$T_{o1} = \frac{3}{2} V_1 - \frac{1}{2} V_8 + \frac{1}{2} \frac{pu}{k} V_{11} + V_{12} + \frac{1}{2} V_{14} \qquad (4\text{-}122)$$

$$T_{o2} = V_2 \qquad (4\text{-}123)$$

$$T_{o3} = \frac{3}{2} V_3 - \frac{1}{2} V_5 - V_{10} - \frac{1}{2} V_{11} \qquad (4\text{-}124)$$

$$T_{o4} = \frac{3}{2} V_4 - \frac{1}{2} V_9 - \frac{1}{2} p^2 V_{11} - V_{13} - \frac{1}{2} pku\, V_{14} \qquad (4\text{-}125)$$

$$T_{o5} = \frac{3}{2} V_5 - \frac{1}{2} V_3 + V_{10} + \frac{1}{2} V_{11} \qquad (4\text{-}126)$$

$$T_{o6} = V_6 \qquad (4\text{-}127)$$

$$T_{o7} = 3V_7 + \frac{1}{2} V_1 + \frac{1}{2} p^2 V_2 + \frac{1}{2} \frac{pu}{k} V_3 + \frac{1}{2} \frac{pu}{k} V_5 + \frac{1}{2} V_6 + \frac{1}{2} V_8 \qquad (4\text{-}128)$$

$$T_{o8} = \frac{3}{2} V_8 - \frac{1}{2} V_1 - \frac{1}{2} \frac{pu}{k} V_{11} - V_{12} - \frac{1}{2} V_{14} \qquad (4\text{-}129)$$

$$T_{o9} = \frac{3}{2} V_9 - \frac{1}{2} V_4 + \frac{1}{2} p^2 V_{11} + V_{13} + \frac{1}{2} pku\, V_{14} \qquad (4\text{-}130)$$

$$T_{o10} = \frac{3}{2} V_{10} - \frac{1}{2} V_3 + \frac{1}{2} V_5 + \frac{1}{2} V_{10} + \frac{1}{2} V_{11} \qquad (4\text{-}131)$$

$$T_{o11} = V_{11} \qquad (4\text{-}132)$$

$$T_{o12} = 2\, V_{12} + \frac{1}{2} V_1 - \frac{1}{2} V_8 + \frac{1}{2} \frac{pu}{k} V_{11} + \frac{1}{2} V_{14} \qquad (4\text{-}133)$$

$$T_{013} = 2\,V_{13} - \frac{1}{2}\,V_4 + \frac{1}{2}\,p^2\,V_{11} + \frac{1}{2}\,pku\,V_{14} + \frac{1}{2}\,V_9 \qquad (4\text{-}134)$$

$$T_{014} = V_{14} \qquad (4\text{-}135)$$

The right-hand side of the vertex equation,

$$-\,\varepsilon\,\frac{\partial}{\partial p_\alpha}\,T^{\lambda\alpha}\,, \qquad (4\text{-}136)$$

is collapsed into a group of eight linearly independent matrix terms by the summation over α. It is clear that in order to obtain the first-order partial derivatives of $T^{\lambda\alpha}$ type functions the first-order partial derivatives of the $W^{\lambda\nu}$ and $Z^{\lambda\nu}$ had to be known. Initial work on this main program tried to avoid this confrontation by seeking the derivatives of $T^{\lambda\alpha}$ by a standard numerical process. This was found unsatisfactory for two reasons. One was the numerical calculation added significantly to the time parameter of the program. The other was the precision which this time bought was inadequate. This method was shortly abandoned in favor of compiling the algebraic forms of the partial derivatives of $T^{\lambda\alpha}$, $W^{\lambda\nu}$ and $Z^{\lambda\nu}$. These are recorded in the program included in Appendix C.

In conclusion the right-hand side of the vertex equation is comprised of eight linearly independent matrix terms. The coefficients of each of these terms are given in Table 4-2.

TABLE 4-2 Right-Hand Side of the Vertex Equations

Equation 1 Coefficient of $p^\lambda \not{p}$

$$\varepsilon[2\,T_{o1}' - \frac{u}{p^2}\,T_{o2}^* + 6\,T_{o2} + 2\,p^2 T_{o2}' + 2\,\frac{p}{k}\,u\,T_{o3}'$$

$$+ \frac{(1-u^2)}{pk}\;T_{o3}^* + 2\,T_{o7}' - \frac{u}{p^2}\,T_{o7}^*].$$

Equation 2 Coefficient of $p^\lambda \not{k}$

$$\varepsilon[T_{o3} + 2\,T_{o4}' - \frac{u}{p^2}\,T_{o4}^* + 5\,T_{o5} + 2\,p^2 T_{o5}'$$

$$+ 2\,pku\,T_{o6}' + (1-u^2)\,\frac{k}{p}\,T_{o6}^* + \frac{k}{p}\,T_{o7}^*].$$

Equation 3 Coefficient of γ^λ

$$\varepsilon[T_{o1} + T_{o7} + 4\,T_{o8} + 2\,p^2\,T_{o8}' + 2\,\frac{p}{k}\,u\,T_{o9}'$$

$$+ \frac{(1-u^2)}{pk}\;T_{o9}^*].$$

Equation 4 Coefficient of $\varepsilon^{\lambda\alpha\beta\phi}\gamma^5\gamma_\phi k_\alpha p_\beta$

$$\varepsilon[T_{o10} + 4\,T_{o11} + 2\,p^2\,T_{o11}' + \frac{k}{p}\,T_{o12}^* + 2\,T_{o13}'$$

$$- \frac{u}{p^2}\;T_{o13}^* - 2\,pku\,T_{o14}' + \frac{k}{p}\,(1-u^2)\,T_{o14}^*].$$

Equation 5 Coefficient of $i\sigma^{\alpha\beta}k_\alpha p_\beta p^\lambda$

$$\varepsilon[6\,T_{e6} + 2\,p^2\,T_{e6}' + 2\,pku\,T_{e7}' + \frac{k}{p}\,(1-u^2)\,T_{e7}^*$$

$$+ 2\,T_{e10}' - \frac{u}{p^2}\,T_{e10}^* + \frac{k}{p}\,T_{e11}^* + 2\,T_{e14}' - \frac{u}{p^2}\,T_{e14}^*].$$

Equation 6 Coefficient of $i\sigma^{\lambda\alpha}p_{\alpha}$

$$\varepsilon[2\, T'_{e4} - \frac{u}{p^2} T^{*}_{e4} - 5\, T_{e8} - 2\, p^2 T'_{e8} - T_{e11}$$

$$- 2\, \frac{pu}{k} T'_{e13} - (1-u^2) \frac{1}{pk} T^{*}_{e13}].$$

Equation 7 Coefficient of $i\sigma^{\lambda\alpha}k_{\alpha}$

$$\varepsilon[\, \frac{k}{p} T^{*}_{e4} + 2\, pku\, T'_{e9} + (1-u^2) \frac{k}{p} T^{*}_{e9} + T_{e10}$$

$$+ 4\, T_{e12} + 2\, p^2 T'_{e12} - T_{e13} + T_{e14}].$$

Equation 8 Coefficient of p^{λ}

$$\varepsilon[5\, T_{e1} + 2\, p^2\, T'_{e1} + 2\, \frac{p}{k}\, u\, T'_{e2} + \frac{(1-u^2)}{pk} T^{*}_{e2} + 2\, T'_{e5}$$

$$- u\, T^{*}_{e5}].$$

CHAPTER V

CHECKING THE ALGEBRA

In performing all the myriad matrix operations necessary to express the vertex equation we found that working out the algebra was extensive, repetitive and subject to error whenever the practitioner's strictest attention lapsed. After an unoriginal effort was made at wading through the whole expression term by term, it was necessary to confirm the results. Faced with the prospect of having to repeat the monumental process, we drew the conclusion that this kind of work is better done by machine. Efforts were made to solicit the use of a computer language capable of symbolic algebra.

The software chosen was a programming language from the University of Utah called Reduce. The program offered a great variety of general algebraic calculating facilities. Amongst those of interest were symbolic differentiation, automatic and user controlled simplification of expressions, calculations with gamma matrices and tensor operations. With all of these capabilities it offered a very promising approach to the unwieldy task.

It was found that Reduce provided a direct and easily acquired check on the left-hand side of the equation. The partial differentiation was performed, including all nine possible mixed derivatives with respect to p^2 and u up to third order. A quick and accurate check of the original

left-hand side was achieved. Furthermore, this confirmation was achieved with only a reasonable expenditure of time invested in becoming familiar with the language.

However Reduce was less easy to make use of on the right-hand side where, in addition to first and second order mixed partial derivatives, there were three layers of matrix operations to undergo. The size of the arrays quickly outgrew the allotted workspace in the machine. All of the calculations had to be performed in steps and then the results were summed afterwards. A further investment of time would have been necessary to learn how to design the output to be displayed in a form more amenable to easy checking. Despite this user related ineptitude, a complete expansion of the right-hand side was obtained.

One of the reasons Reduce was found to be less useful than expected on the right-hand side was the operations, though limited in variety, involved a proliferation of terms. It was almost a waste to bring all of the ingenious operational flexibility of Reduce to bear on what was only a problem of tensor and matrix multiplication and large-scale sorting of terms.

The right-hand side could be written down in a way that was better designed for checking by giving up the luxury of exhibiting the right-hand side in terms of the basic functions A, B, F, G_0, G_2, H_0, H_1, H_2 and I. The final form settled upon for the right-hand side, as it appeared in the previous chapter, was expressed in a hierarchy of definitions. The right-hand side was given in terms of the 28 components of $T^{\lambda\alpha}$ tensor. The 28 components of $T^{\lambda\alpha}$ were given in terms of the 28

components of the $F^{\lambda\nu}$ tensor. The $F^{\lambda\nu}$ tensor was defined in terms of the 28 components of the $Z^{\lambda\nu}$ tensor. Finally the 28 components of the $Z^{\lambda\nu}$ tensor were defined in terms of the basic functions, A, B, F, G_0, G_1, G_2, H_0, H_1, H_2 and I. Each layer of redefinition represents the execution of another tensor operation.

Table 5-1 summarizes the five steps taken to define the right-hand side. The first step was to define a tensor,

$$Z^{\lambda\nu} = -2PPD\, \Gamma^{\lambda} p^{\nu} - ABD\, \Gamma^{\lambda}\gamma^{\nu} + BBD\, \Gamma^{\lambda}\not{p}\gamma^{\nu} + 2MPD\, \Gamma^{\lambda}\not{p}p^{\nu}. \tag{5-1}$$

The abbreviations PPD, ABD, BBD and MPD, represent combinations of the electron functions, A and B, which were defined in Eqs. (4-39), (4-40), (4-41) and (4-42). Step 2 forms a new tensor,

$$W^{\lambda\nu} = (\not{p} + \not{k}) Z^{\lambda\nu}. \tag{5-2}$$

Step 3 forms yet another tensor out of the former two tensors,

$$F^{\lambda\nu} = \frac{A_1}{D_1} Z^{\lambda\nu} - \frac{B_1}{D_1} W^{\lambda\nu} \tag{5-3}$$

where $D_1 = (A_1^2 - p_1^2 B_1^2)$.

Step 4 forms the last tensor,

$$T^{\lambda\alpha} = [g^{\alpha}_{\nu} + \frac{1}{2}\gamma^{\alpha}\gamma_{\nu}] F^{\lambda\nu}. \tag{5-4}$$

Finally, in Step 5, the α index is contracted by a differentiation with respect to p^{α}. The right-side equals

$$-\varepsilon \frac{\partial}{\partial p^{\alpha}} T^{\lambda\alpha}. \tag{5-5}$$

TABLE 5-1 Formation of the Right-Hand Side of the Vertex Equation

Step 1.[†] $Z^{\lambda\nu} = -2PPD\ \Gamma^{\lambda} p^{\nu} - ABD\ \Gamma^{\lambda} \gamma^{\nu} + BBD\ \Gamma^{\lambda} \not{p} \gamma^{\nu}$

$+ 2MPD\ \Gamma^{\lambda} \not{p} p^{\nu}$

Step 2. $W^{\lambda\nu} = (\not{p} + \not{k}) Z^{\lambda\nu}$

Step 3.[†] $F^{\lambda\nu} = \frac{A_1}{D_1} Z^{\lambda\nu} - \frac{B_1}{D_1} W^{\lambda\nu}$

Step 4. $T^{\lambda\alpha} = [g^{\alpha}_{\nu} + \frac{1}{2} \gamma^{\alpha} \gamma_{\nu}] F^{\lambda\nu}$

Step 5. Right-hand side $= -\varepsilon \frac{\partial}{\partial p^{\alpha}} T^{\lambda\alpha}$

[†]The abbreviations PPD, BBD, ABD, and MPD were defined in Eqs. (4-39) through (4-42). The abbreviations A_1 and D_1 were defined in Eq. (4-24)

After the right-hand side is separated into layers so that the full length and breadth of the right-hand side is disguised, a large number of operations still have to be performed. Even a simple multiplication like $\Gamma^{\lambda}\not{p}$ or $(\not{p}+\not{k})Z^{\lambda\nu}$ involves a large number of steps. Consider the procedure necessary to perform the multiplication of a single element of Γ^{λ}, say $\varepsilon^{\lambda\alpha\beta\phi}\,\gamma^5\gamma_{\phi}k_{\alpha}p_{\beta}\,\frac{I}{pk}$, with $\not{p}$.

$$\varepsilon^{\lambda\alpha\beta\phi}\,\gamma^5\gamma_{\phi}k_{\alpha}p_{\beta}\,\frac{I}{pk}\,\not{p} = \varepsilon^{\lambda\alpha\beta\phi}\,k_{\alpha}p_{\beta}p^{\eta}\gamma^5\gamma_{\phi}\gamma_{\eta}\,\frac{I}{pk}$$

$$= \varepsilon^{\lambda\alpha\beta\phi}\,k_{\alpha}p_{\beta}p^{\eta}[g_{\phi\eta}\gamma^5 + \frac{i}{2}\,\varepsilon_{\phi\eta\xi\tau}\sigma^{\xi\tau}]\,\frac{I}{pk}$$

$$= -\frac{i}{2}\,\varepsilon^{\lambda\alpha\beta\phi}\,\varepsilon_{\eta\xi\tau\phi}\,k_{\alpha}p_{\beta}p^{\eta}\,\sigma^{\xi\tau}\,\frac{I}{pk}$$

$$= -\frac{i}{2}[2g^{\lambda}_{\eta}\sigma^{\beta\alpha} + 2g^{\alpha}_{\eta}\sigma^{\lambda\beta} + 2g^{\beta}_{\eta}\sigma^{\alpha\lambda}]k_{\alpha}p_{\beta}p^{\eta}\,\frac{I}{pk}$$

$$= -\,[i\sigma^{\beta\gamma}k_{\alpha}p_{\beta}p^{\lambda} + ipku\,\sigma^{\lambda\alpha}p_{\alpha} + ip^2\sigma^{\alpha\lambda}k_{\alpha}]\,\frac{I}{pk}. \quad (5\text{-}6)$$

Table A-1 was used to obtain the product $\gamma^5\gamma_{\phi}\gamma_{\eta}$ in the second line. Equation A-12 was used to go from the third to fourth line.

It is easy to forget a sign or reverse the order of two indices when a large number of such operations are performed. However it is possible to express the multiplication rules for matrix operations in a very simple way that allows the practitioner to do the same calculation in his head without ever consulting Table A-1. This simplified method can be used to supply an algorithm to enable a computer to do the same kinds of manipulations in a common language like Fortran which does not possess symbolic capabilities. The basis of this method was designed by H. S. Green.

To explain the simple multiplication method it is necessary to alter slightly the form of the definitions of vector functions like Γ^{λ} and tensor functions like $Z^{\lambda\nu}$. The expressions used in Chapter IV were evolved through a historical process that did not necessarily produce the most symmetric arrangement. In this chapter it will be shown that when some small changes are made, things become very much easier.

In Chapter IV,

$$\Gamma^{\lambda} = p^{\lambda}\frac{F}{pk} + \gamma^{\lambda}G_o + p^{\lambda}\not{p}\frac{G_1}{p^2} + p^{\lambda}\not{k}\frac{G_2}{pk}$$

$$+ 2p^{\lambda}i\sigma^{\alpha\beta}k_{\alpha}p_{\beta}\frac{H_o}{p^2k^2} + 2i\sigma^{\alpha\lambda}p_{\alpha}\frac{H_1}{pk} + 2i\sigma^{\alpha\lambda}k_{\alpha}\frac{H_2}{pk}$$

$$+ \varepsilon^{\lambda\alpha\beta\phi}\gamma^5\gamma_{\phi}\, p_{\beta}k_{\alpha}\frac{I}{pk}\,. \qquad (5\text{-}7)$$

Now a slightly altered form will be used.

$$\Gamma^{\lambda} = p^{\lambda}\ell_1 + \gamma^{\lambda}\ell_2 + p^{\lambda}\not{p}\ell_3 + p^{\lambda}\not{k}\ell_4$$

$$+ p^{\lambda}i\sigma^{\alpha\beta}k_{\alpha}p_{\beta}\ell_5 + i\sigma^{\lambda\alpha}p_{\alpha}\ell_6 + i\sigma^{\lambda\alpha}k_{\alpha}\ell_7$$

$$+ \varepsilon^{\lambda\alpha\beta\phi}\gamma^5\gamma_{\phi}k_{\alpha}p_{\beta}\ell_8 \qquad (5\text{-}8)$$

where

$$\ell_1 = \frac{F}{pk} \qquad \ell_5 = \frac{2H_o}{p^2k^2}$$

$$\ell_2 = G_o \qquad \ell_6 = -\frac{2H_1}{pk}$$

$$\ell_3 = \frac{G_1}{p^2} \qquad \ell_7 = -\frac{2H_2}{k}$$

$$\ell_4 = \frac{G_2}{pk} \qquad \ell_8 = \frac{I}{pk}\,. \qquad (5\text{-}9)$$

The new form of Γ^λ is equivalent but notice that the order of the indices in the sixth and seventh components has been changed. Throughout this chapter it will be assumed that any general vector, say V^λ, is expanded in terms of these same components.

$$V^\lambda = p^\lambda v_1 + \gamma^\lambda v_2 + p^\lambda \not{p} v_3 + p^\lambda \not{k} v_4 + p^\lambda i\sigma^{\alpha\beta}k_\alpha p_\beta v_5 + i\sigma^{\lambda\alpha}p_\alpha v_6 + i\sigma^{\lambda\alpha}k_\alpha v_7 + \varepsilon^{\lambda\alpha\beta\phi}\gamma^5\gamma_\phi k_\alpha p_\beta v_8. \tag{5-10}$$

Similarly it will be required that any general tensor, $T^{\lambda\nu}$, will be expressed in terms of the following components only.

$$\begin{aligned} T^{\lambda\nu} = {} & p^\lambda p^\nu t_1 + p^\lambda p^\nu \not{p} t_2 + p^\lambda p^\nu \not{k} t_3 + p^\lambda p^\nu i\sigma^{\alpha\beta}k_\alpha p_\beta t_4 \\ & + p^\nu \gamma^\lambda t_5 + p^\nu i\sigma^{\lambda\alpha}p_\alpha t_6 + p^\nu i\sigma^{\lambda\alpha}k_\alpha t_7 \\ & + p^\nu \varepsilon^{\lambda\alpha\beta\phi}\gamma^5\gamma_\phi k_\alpha p_\beta t_8 + p^\lambda \gamma^\nu t_9 + p^\lambda i\sigma^{\nu\alpha}p_\alpha t_{10} \\ & + p^\lambda i\sigma^{\nu\alpha}k_\alpha t_{11} + p^\lambda \varepsilon^{\nu\alpha\beta\phi}\gamma^5\gamma_\phi k_\alpha p_\beta t_{12} \\ & + i\sigma^{\nu\lambda}t_{13} + \varepsilon^{\lambda\alpha\nu\phi}\gamma^5\gamma_\phi p_\alpha t_{14} + \varepsilon^{\lambda\alpha\nu\phi}\gamma^5\gamma_\phi k_\alpha t_{15} \\ & + \varepsilon^{\lambda\nu\alpha\beta}\gamma^5 k_\alpha p_\beta t_{16} + p^\lambda k^\nu t_{17} + p^\lambda k^\nu \not{p} t_{18} + p^\lambda k^\nu \not{k} t_{19} \\ & + p^\lambda k^\nu i\sigma^{\alpha\beta}k_\alpha p_\beta t_{20} + \gamma^\lambda k^\nu t_{21} + k^\nu i\sigma^{\lambda\alpha}p_\alpha t_{22} \\ & + k^\nu i\sigma^{\lambda\alpha}k^\alpha t_{23} + k^\nu \varepsilon^{\lambda\alpha\beta\phi}\gamma^5\gamma_\phi k_\alpha p_\beta t_{24} \\ & + g^{\lambda\nu}t_{25} + g^{\lambda\nu}\not{p} t_{26} + g^{\lambda\nu}\not{k} t_{27} + g^{\lambda\nu}i\sigma^{\alpha\beta}k_\alpha p_\beta t_{28}. \end{aligned} \tag{5-11}$$

Now that the groups of matrices have been carefully selected for any general vector V^{λ} or tensor $T^{\lambda\nu}$, the following notation will prove extremely useful. The components of a vector will be denoted by the following brackets.

$$p^{\lambda} = (p^{\lambda}, 0, 0, 0)$$

$$p^{\lambda}\not{p} = (p^{\lambda}\not{p}, 0, 0, 0)$$

$$p^{\lambda}\not{k} = (p^{\lambda}, \not{k}, 0, 0)$$

$$p^{\lambda}i\sigma^{\alpha\beta}k_{\alpha}p_{\beta} = (p^{\lambda}\not{p}, \not{k}, 0, 0)$$

$$\gamma^{\lambda} = (0, 0, \gamma^{\lambda}, 0)$$

$$i\sigma^{\lambda\alpha}p_{\alpha} = (\not{p}, 0, \gamma^{\lambda}, 0)$$

$$i\sigma^{\lambda\alpha}k_{\alpha} = (0, \not{k}, \gamma^{\lambda}, 0)$$

$$\varepsilon^{\lambda\alpha\beta\phi}\gamma^{5}\gamma_{\phi}k_{\alpha}p_{\beta} = (\not{p}, \not{k}, \gamma^{\lambda}, 0). \tag{5-12}$$

The components of a tensor, $T^{\lambda\nu}$, will be denoted by the following brackets.

$$p^{\lambda}p^{\nu} = (p^{\lambda}p^{\nu}, 0, 0, 0)$$

$$p^{\lambda}p^{\nu}\not{p} = (p^{\lambda}p^{\nu}\not{p}, 0, 0, 0)$$

$$p^{\lambda}p^{\nu}\not{k} = (p^{\lambda}p^{\nu}\not{k}, 0, 0, 0)$$

$$p^{\lambda}p^{\nu}i\sigma^{\alpha\beta}k_{\alpha}p_{\beta} = (p^{\lambda}p^{\nu}\not{p}, \not{k}, 0, 0)$$

$$p^{\nu}\gamma^{\lambda} = (p^{\nu},\ 0,\ \gamma^{\lambda},\ 0)$$

$$p^{\nu}i\sigma^{\lambda\alpha}p = (p^{\nu}\not{p},\ 0,\ \gamma^{\lambda},\ 0)$$

$$p^{\nu}i\sigma^{\lambda\alpha}k_{\alpha} = (p^{\nu},\ \not{k},\ \gamma^{\lambda},\ 0)$$

$$p^{\nu}\varepsilon^{\lambda\alpha\beta\phi}\gamma^{5}\gamma_{\phi}k_{\alpha}p_{\beta} = (p^{\nu}\not{p}\ ,\ \not{k},\ \gamma^{\lambda},\ 0)$$

$$p^{\lambda}\gamma^{\nu} = (p^{\lambda},\ 0,\ 0,\ \gamma^{\nu})$$

$$p^{\lambda}i\sigma^{\nu\alpha}p_{\alpha} = (p^{\lambda}\not{p},\ 0,\ 0,\ \gamma^{\nu})$$

$$p^{\lambda}i\sigma^{\nu\alpha}k_{\alpha} = (p^{\lambda},\ \not{k},\ 0,\ \gamma^{\nu})$$

$$p^{\lambda}\varepsilon^{\nu\alpha\beta\phi}\gamma^{5}\gamma_{\phi}k_{\alpha}p_{\beta} = (p^{\lambda}\not{p},\ \not{k},\ 0,\ \gamma^{\nu})$$

$$i\sigma^{\nu\lambda} = (0,\ 0,\ \gamma^{\lambda},\ \gamma^{\nu})$$

$$\varepsilon^{\lambda\alpha\nu\phi}\gamma^{5}\gamma_{\phi}p_{\alpha} = (\not{p},\ 0,\ \gamma^{\lambda},\ \gamma^{\nu})$$

$$\varepsilon^{\lambda\alpha\nu\phi}\gamma^{5}\gamma_{\phi}k_{\alpha} = (0,\ \not{k},\ \gamma^{\lambda},\ \gamma^{\nu})$$

$$\varepsilon^{\lambda\nu\alpha\beta}\gamma^{5}k_{\alpha}p_{\beta} = (\not{p},\ \not{k},\ \gamma^{\lambda},\ \gamma^{\nu})$$

$$i\sigma^{\nu\lambda} = (0,\ 0,\ \gamma^{\lambda},\ \gamma^{\nu})$$

$$\varepsilon^{\lambda\alpha\nu\phi}\gamma^{5}\alpha_{\phi}p_{\alpha} = (\not{p},\ 0,\ \gamma^{\lambda},\ \gamma^{\nu})$$

$$\varepsilon^{\lambda\alpha\nu\phi}\gamma^{5}\gamma_{\phi}k_{\alpha} = (0,\ \not{k},\ \gamma^{\lambda},\ \gamma^{\nu})$$

$$\varepsilon^{\lambda\nu\alpha\beta}\gamma^{5}k_{\alpha}p_{\beta} = (\not{p},\ \not{k},\ \gamma^{\lambda},\ \gamma^{\nu})$$

$$p^\lambda k^\nu = (p^\lambda,\ k^\nu,\ 0,\ 0)$$

$$p^\lambda k^\nu \not{p} = (p^\lambda \not{p},\ k^\nu,\ 0,\ 0)$$

$$p^\lambda k^\nu \not{k} = (p^\lambda,\ k^\nu \not{k},\ 0,\ 0)$$

$$p^\lambda k^\nu i\sigma^{\alpha\beta} k_\alpha p_\beta = (p^\lambda \not{p},\ k^\nu \not{k},\ 0,\ 0)$$

$$\gamma^\lambda k^\nu = (0,\ k^\nu,\ \gamma^\lambda,\ 0)$$

$$k^\nu i\sigma^{\lambda\alpha} p_\alpha = (\not{p},\ k^\nu,\ \gamma^\lambda,\ 0)$$

$$k^\nu i\sigma^{\lambda\alpha} k_\alpha = (0,\ k^\nu \not{k},\ \gamma^\lambda,\ 0)$$

$$k^\lambda \varepsilon^{\lambda\alpha\beta\phi} \gamma^5 \gamma_\phi k_\alpha p_\beta = (\not{p},\ k^\nu \not{k},\ \gamma^\lambda,\ 0)$$

$$g^{\lambda\nu} = (0,\ 0,\ g^{\lambda\nu},\ 0)$$

$$g^{\lambda\nu} \not{p} = (\not{p},\ 0,\ g^{\lambda\nu},\ 0)$$

$$g^{\lambda\nu} \not{k} = (0,\ \not{k},\ g^{\lambda\nu},\ 0)$$

$$g^{\lambda\nu} i\sigma^{\alpha\beta} \not{k}_\alpha p_\beta = (\not{p},\ \not{k},\ g^{\lambda\nu},\ 0). \tag{5-13}$$

The rule for the mapping of each component into the bracket notation is straightforward. The first position in each bracket holds all electron momenta terms; p^λ, p^ν and $\not{p}$. The second position holds all photon momenta terms, k^ν and $\not{k}$. (The k^λ term does not appear here because only the transverse part of the vertex equation has been considered. However it is possible to extend the bracket notation to include longitudinal components without any additional complications.) The third position holds γ^λ and $g^{\lambda\nu}$. The fourth and last position holds γ^ν.

Now to convert the component into its bracket merely tabulate the occurrence of $\gamma^{\beta}p_{\beta}$, $\gamma^{\alpha}k_{\alpha}$, γ^{λ}, γ^{ν}, p^{λ}, p^{ν}, k^{ν} and $g^{\mu\nu}$ and put them into their appropriate "home" positions. For example, consider $p^{\lambda}i\sigma^{\nu\alpha}k_{\alpha}$. It contains p^{λ}, γ^{ν} and $k_{\alpha}\gamma^{\alpha}$ so its bracket is $(p^{\lambda}, \not{k}, 0, \gamma^{\nu})$. Notice that although there is a unique bracket notation for each component, it is not necessarily possible to guess from the bracket notation what the original choice was for the component.

$$(\not{p}, \not{k}, 0, 0) = i\sigma^{\alpha\beta}p_{\alpha}k_{\beta}$$

or

$$(\not{p}, \not{k}, 0, 0) = i\sigma^{\alpha\rho}k_{\alpha}p_{\beta}?$$

The originally selected meaning of $(\not{p}, \not{k}, 0, 0)$ must be preserved so it is the latter relationship which is the correct one.

Now it will be shown that multiplying the components of the vector V^{λ} or tensor $T^{\lambda\nu}$ by a term like $\not{p}$, $\not{k}$, γ_{ν} p_{ν} ,k_{ν}, etc. is simply done by observing a few rules. To multiply by a unit matrix vector like p_{ν} or k_{ν} merely add the vector to its correct position.

$$(\not{p}, \not{k}, \gamma^{\lambda}, 0)p^{\nu} = (p^{\nu}\not{p}, \not{k}, \gamma^{\lambda}, 0) \tag{5-14}$$

To multiply by a matrix like γ^{ν}, $\not{p}$ or $\not{k}$ from the right, move the matrix across the bracket from right to left. Each time the matrix crosses another matrix take the scalar product of the two. Remove the scalar products to the outside of the brackets.

$$(p^{\lambda}\not{p}, p, 0, 0)\not{p} = p^{2}(p^{\lambda}, 0, 0, 0). \tag{5-15}$$

One term will occur for each scalar product and one term will occur when the multiplying gamma matrix reaches its "home" position,

$$(p^\lambda \not{p},\ k^\nu,\ 0,\ 0)\not{k} = (p^\lambda \not{p},\ k^\nu\ \ \not{k},\ 0,\ 0) + pku(p^\lambda,\ k^\nu,\ 0,\ 0). \qquad (5\text{-}16)$$

Finally, reverse the sign of alternate scalar products,

$$\begin{aligned}(p^\lambda \not{p},\ \not{k},\ 0,\ \gamma^\nu)\not{k} = (p^\lambda \not{p},\ k^\nu \not{k},\ 0,\ 0) &- k^2(p^\lambda \not{p},\ 0,\ 0,\ \gamma^\nu) \\ &+ pku(p^\lambda,\ \not{k},\ 0,\ \gamma^\nu). \qquad (5\text{-}17)\end{aligned}$$

In order to multiply a tensor or vector from the left the rules remain the same except that the multiplying term is moved across the bracket from left to right.

Recall the example given earlier of the multiplication of $\varepsilon^{\lambda\alpha\beta\phi}\gamma^5_\gamma\gamma_\phi k_\alpha p_\beta \frac{I}{pk}$ by $\not{p}$ from the right. This example can now be written as

$$\begin{aligned}(\not{p},\ \not{k},\ \gamma^\lambda,\ 0)\ \frac{I}{pk}\ \not{p} = [(p^\lambda \not{p},\ \not{k},\ 0,\ 0) &- pku(\not{p},\ 0,\ \gamma^\lambda,\ 0) \\ &+ p^2(0,\ \not{k},\ \gamma^\lambda,\ 0)]\ \frac{I}{pk}. \qquad (5\text{-}18)\end{aligned}$$

Translating the bracket notation back to the original components it is found that

$$\begin{aligned}\varepsilon^{\lambda\alpha\beta\phi}\gamma^5_\gamma\gamma_\phi k_\alpha p_\beta \frac{I}{pk} = [p^\lambda i\sigma^{\alpha\beta}k_\alpha p_\beta &- pku\ i\sigma^{\lambda\alpha}p_\alpha \\ &+ p^2 i\sigma^{\lambda\alpha}k_\alpha]\ \frac{I}{pk}\ . \qquad (5\text{-}19)\end{aligned}$$

This is the same result which was achieved in Eq. (5-6), yet it was achieved without using Table A-1. With little practice the products can be arrived at as quickly as they can be written down. This process

of bracket manipulation is not only many times easier for the human mind, but it also lends itself to the writing of an algorithm for executing similar products in Fortran.

It is also possible to predict easily the products of a contraction of a vector V^α or tensor $T^{\lambda\alpha}$ with another vector like p_α or gamma matrix like γ_α. To contract with a momentum, merely perform the implied contraction. To contract with a gamma matrix it is necessary to first move the gamma matrix across the bracket, taking alternate signs of each possible scalar product. While doing so, perform the implied contractions on each term in the product.

A summary of the rules for bracket operations is given in Table 5-2.

Returning to Table 5-1 it is easy to see that a knowledge of the following operations is all that is needed to generate the right-hand side, $V^\lambda \not{p}$, $V^\lambda p^\nu$, $V^\lambda \gamma^\nu$, $\gamma^\nu V^\lambda$, $\not{k}\, T^{\lambda\nu}$, $\not{p}\, T^{\lambda\nu}$ and $\gamma_\nu T^{\lambda\nu}$, where V^λ stands for any general vector function and $T^{\lambda\nu}$ stands for any general tensor function. Using the expressions for V^λ and $T^{\lambda\nu}$ given in Eqs. (5-10) and (5-11) and using the method of bracket operations, these seven products are easily written down.

$$
\begin{aligned}
(1)\quad V^\lambda \not{p} = {} & p^\lambda(p^2 V_2 + pku\, v_3 + v_5) + (p^\lambda \not{p}(v_1 + pku\, v_4 + v_6) \\
& + p^\lambda k(-p^2 v_4 + v_7) + p^\lambda i\sigma^{\alpha\beta} k_\alpha p_\beta (v_8 - v_3) \\
& + \gamma^\lambda(-p^2 v_6 - pku\, v_7) + i\sigma^{\lambda\alpha} p_\alpha(-v_5 - pku\, v_8) \\
& + i\sigma^{\lambda\alpha} k_\alpha (p^2 v_8) + \varepsilon^{\lambda\alpha\beta\phi}\gamma^5\gamma_\phi k_\alpha p_\beta (v_7)
\end{aligned}
\tag{5-20}
$$

TABLE 5-2 Bracket Operations

A. Multiplying by a momentum: Add the momentum to its "home" position.

B. Contracting with a momentum: Merely contract the momentum with its proper complement and move the scalar product to the outside of the bracket.

C. Multiplying by a gamma matrix:

(i) Move the matrix across the bracket from right to left in order to multiply from the right. Reverse directions to multiply from the left.

(ii) Take alternate signs of each scalar product that can be formed.

(iii) If there is no matrix in "home" position, include a term with the multiplying matrix in the "home" position.

D. Contracting with a gamma matrix:

(i) Perform steps (i), (ii).

(ii) Perform the implied contraction as the gamma matrix is moved across the bracket.

$$(2)\quad V^{\lambda}p^{\nu} = p^{\lambda}p^{\nu}(v_1) + p^{\lambda}p^{\nu}\not{p}(v_2) + p^{\lambda}p^{\nu}\not{k}(v_3) + p^{\lambda}p^{\nu}i\sigma^{\alpha\beta}k_{\alpha}p_{\beta}(v_4)$$

$$+ p^{\nu}p^{\lambda}(v_5) + p^{\nu}i\sigma^{\lambda\alpha}p_{\alpha}(v_6) + p^{\nu}i\sigma^{\lambda\alpha}k_{\alpha}(v_8) + p^{\nu}\varepsilon^{\lambda\alpha\beta\phi}\gamma^{5}\gamma_{\phi}k_{\alpha}p_{\beta}(v_8) \tag{5-21}$$

$$(3)\quad V^{\lambda}\gamma^{\nu} = p^{\lambda}p^{\nu}(v_2) + p^{\lambda}p^{\nu}\not{k}(-v_4) + p^{\nu}\gamma^{\lambda}(-v_6) + p^{\nu}i\sigma^{\lambda\alpha}k_{\alpha}(v_8)$$

$$+ p^{\lambda}\gamma^{\nu}(v_1) + p^{\lambda}i\sigma^{\nu\alpha}p_{\alpha}(v_2) + p^{\lambda}i\sigma^{\nu\alpha}k_{\alpha}(v_3)$$

$$+ p^{\lambda}\varepsilon^{\nu\alpha\beta\phi}\gamma^{5}\gamma_{\phi}k_{\alpha}p_{\beta}(v_4) + i\sigma^{\nu\lambda}(v_5) + \varepsilon^{\lambda\alpha\nu\phi}\gamma^{5}\gamma_{\phi}p_{\alpha}(v_6)$$

$$+ p^{\lambda}k^{\nu}(v_3) + p^{\lambda}k^{\nu}\not{p}(v_4) + p^{\lambda}k^{\nu}\not{k}(v_7) + p^{\lambda}k^{\nu}i\sigma^{\alpha\beta}k_{\alpha}p_{\beta}(v_8)$$

$$+ \gamma^{\lambda}k^{\nu}(-v_7) + k^{\nu}i\sigma^{\lambda\alpha}p_{\alpha}(-v_8) + g^{\lambda\nu}(v_5) + g^{\lambda\nu}\not{p}(v_6)$$

$$+ g^{\lambda\nu}\not{k}(v_7) + g^{\lambda\nu}i\sigma^{\alpha\beta}k_{\alpha}p_{\beta}(v_8) \tag{5-22}$$

$$(4)\quad \gamma^{\nu}V^{\lambda} = p^{\lambda}p^{\nu}(v_2) + p^{\lambda}p^{\nu}\not{k}(v_4) + p^{\nu}\gamma^{\lambda}(v_6) + p^{\nu}i\sigma^{\lambda\alpha}k_{\alpha}(v_8)$$

$$+ p^{\lambda}\gamma^{\nu}(v_1) + p^{\lambda}i\sigma^{\nu\alpha}p_{\alpha}(-v_2) + p^{\lambda}i\sigma^{\nu\alpha}k_{\alpha}(-v_3)$$

$$+ p^{\lambda}\varepsilon^{\nu\alpha\beta\phi}\gamma^{5}\gamma_{\phi}k_{\alpha}p_{\beta}(v_4) + i\sigma^{\nu\lambda}(-v_5) + \varepsilon^{\lambda\alpha\nu\phi}\gamma^{5}\gamma_{\phi}p_{\alpha}(v_6)$$

$$+ \varepsilon^{\lambda\alpha\nu\phi}\gamma^{5}\gamma_{\phi}k_{\alpha}(v_7) + \varepsilon^{\lambda\nu\alpha\beta}\gamma^{5}p_{\beta}k_{\alpha}(-v_8) + p^{\lambda}k^{\nu}(v_3)$$

$$+ p^{\lambda}k^{\nu}\not{p}(-v_4) + \gamma^{\lambda}k^{\nu}(v_7) + k^{\nu}i\sigma^{\lambda\alpha}p_{\alpha}(-v_8) + g^{\lambda\nu}(v_5)$$

$$+ g^{\lambda\nu}\not{p}(-v_6) + g^{\lambda\nu}\not{k}(-v_7) + g^{\lambda\nu}i\sigma^{\alpha\beta}k_{\alpha}p_{\beta}(v_8) \tag{5-23}$$

$$(5)\quad \not{k}T^{\lambda\nu} = p^{\lambda}p^{\nu}(pku\ t_2 + k^2t_3) + p^{\lambda}p^{\nu}\not{p}(-k^2t_4)$$

$$+ p^{\lambda}p^{\nu}\not{k}(t_1 + pku\ t_4) + p^{\lambda}p^{\nu}i\sigma^{\alpha\beta}k_{\alpha}p_{\beta}(-t_2)$$

$$+ p^\nu \gamma^\lambda (pku\ t_6 + k^2 t_7) + p^\nu i\sigma^{\lambda\alpha} p_\alpha (-k^2 t_8)$$

$$+ p^\nu i\sigma^{\lambda\alpha} k_\alpha (t_5 + pku\ t_8) + p^\nu \varepsilon^{\lambda\alpha\beta\phi} \gamma^5 \gamma_\phi k_\alpha p_\beta (-t_6)$$

$$+ p^\lambda \gamma^\nu (pku\ t_{10} + k^2 t_{11}) + p^\lambda i\sigma^{\nu\alpha} p_\alpha (-k^2 t_{12})$$

$$+ p^\lambda i\sigma^{\nu\alpha} k_\alpha (t_9 + pku\ t_{12}) + p^\lambda \varepsilon^{\nu\alpha\beta\phi} \gamma^5 \gamma_\phi k_\alpha p_\beta (-t_{10})$$

$$+ i\sigma^{\nu\lambda} (pku\ t_{19} + k^2\ t_{15}) + \varepsilon^{\lambda\alpha\nu\phi} \gamma^5 \gamma_\phi p_\alpha (-k^2 t_{16})$$

$$+ \varepsilon^{\lambda\alpha\nu\phi} \gamma^5 \gamma_\phi k_\alpha (t_{13} + pku\ t_{16}) + \varepsilon^{\lambda\nu\alpha\beta} \gamma^5 p_\beta k_\alpha (-t_{14})$$

$$+ p^\lambda k^\nu (t_9 + pku\ t_{18} + k^2 t_{19}) + p^\lambda k^\nu \not{p} (-t_{10} - k^2 t_{20})$$

$$+ p^\lambda k^\nu \not{k} (t_{17} + pku\ t_{20} - t_{11}) + p^\lambda k^\nu i^{\alpha\beta}\ k_\alpha p_\beta (t_{12} - t_{18})$$

$$+ \gamma^\lambda k^\nu (-t_{13} + pku\ t_{22} + k^2 t_{23}) + k^\nu i\sigma^{\lambda\alpha} p_\alpha (t_{14} - k^2 t_{24})$$

$$+ k^\nu i\sigma^{\lambda\nu} k\ (t_{15} + t_{21} + pku\, t_{24}) + k^\nu \varepsilon^{\lambda\alpha\beta\phi} \gamma^5 \gamma_\phi k_\alpha p_\beta (-5_{16} - t_{22})$$

$$+ g^{\lambda\nu} (pku\ t_{26} + k^2 t_{27}) + g^{\lambda\nu} \not{p} (-k^2 t_{28})$$

$$+ g^{\lambda\nu} \not{k} (t_{25} + pku\ t_{28}) + g^{\lambda\nu} i\sigma^{\alpha\beta} k_\alpha p_\beta (-t_{26}) \qquad (5\text{-}24)$$

(6) $$\not{p} T^{\lambda\nu} = p^\lambda \not{p}\ (p^2 t_2 + t_5 + t_9 + pku\ t_3)$$

$$+ p^\lambda p^\nu \not{p} (t_1 - pku\ t_4 - t_6 - t_{10})$$

$$+ p^\lambda p^\nu \not{k} (p^2 t_4 - t_7 - t_{11})$$

$$+ p^\lambda p^\nu i\sigma^{\alpha\beta} k_\alpha p_\beta (t_3 + t_8 + t_{12})$$

$$+ p^\nu \gamma^\lambda (p^2 t_6 + pku\ t_7 - t_{13})$$

$$+ p^{\nu} i\sigma^{\lambda\alpha} p_{\alpha}(t_5 - pku\ t_8 + t_{14})$$

$$+ p^{\nu} i\sigma^{\lambda\alpha} k_{\alpha}(p^2 t_8 + t_{15})$$

$$+ p^{\nu} \varepsilon^{\lambda\alpha\beta\phi} \gamma^5 \gamma_{\phi} k_{\alpha} p_{\beta}(t_7 - t_{16})$$

$$+ p^{\lambda}\ {}^{\nu}(p^2\ t_{10} + t_{13} + pku\ t_{11})$$

$$+ p^{\lambda} i\sigma^{\nu\alpha} p_{\alpha}(t_9 - pku\ t_{12} - t_{14})$$

$$+ p^{\lambda} i\sigma^{\nu\alpha} k_{\alpha}(p^2 t_{12} + t_{15})$$

$$+ p^{\lambda} \varepsilon^{\nu\alpha\beta\phi} \gamma^5 \gamma_{\phi} k_{\alpha} p_{\beta}(t_{11} + t_{16})$$

$$+ i\sigma^{\nu\lambda}(p^2 t_{14} + pku\ t_{15})$$

$$+ \varepsilon^{\lambda\alpha\nu\phi} \gamma^5 \gamma_{\phi} p_{\alpha}(t_{13} - pku\ t_{16})$$

$$+ \varepsilon^{\lambda\alpha\nu\phi} \gamma^5 \gamma_{\phi} k_{\alpha}(p^2\ t_{16})$$

$$+ \varepsilon^{\lambda\nu\alpha\beta} \gamma^5 p_{\beta} k_{\alpha}(t_{15})$$

$$+ p^{\lambda} k^{\nu}(p^2 t_{18} + pku\ t_{19} + t_{21})$$

$$+ p^{\lambda} k^{\nu} \not{p})t_{17} - pku\ t_{20} - t_{22})$$

$$+ p^{\lambda} k^{\nu} \not{k}(p^2 t_{20} - t_{23})$$

$$+ p^{\lambda} k^{\nu} i\sigma^{\alpha\beta} k_{\alpha} p_{\beta}(t_{19} + t_{24})$$

$$+ \gamma^{\lambda} k^{\nu}(p^2 t_{22} + pku\ t_{23})$$

$$+ k^{\nu} i\sigma^{\lambda\alpha} p_{\alpha}(t_{21} - pku\ t_{24})$$

$$+ k^{\nu} i\sigma^{\lambda\alpha} k_{\alpha} (p^2 t_{24})$$

$$+ k^{\nu} \varepsilon^{\lambda\alpha\beta\phi} \gamma^5 \gamma_{\phi} k_{\alpha} p_{\beta} (t_{23})$$

$$+ g^{\lambda\nu} (p^2 t_{26} + \text{pku } t_{27})$$

$$+ g^{\lambda\nu} \not{p} (t_{25} - \text{pku } t_{28})$$

$$+ g^{\lambda\nu} \not{k} (p^2 t_{28})$$

$$+ g^{\lambda\nu} i\sigma^{\alpha\beta} k_{\alpha} p_{\beta} (t_{27}) \tag{5-25}$$

(7) $$\gamma_{\nu} T^{\lambda\nu} = p^{\lambda} (p^2 t_2 + \text{pku } t_3 + t_5 + 4t_9 + \text{pku } t_{18} + k^2 t_{19} + t_{26})$$

$$+ p^{\lambda} \not{p} (t_1 - \text{pku } t_4 - t_5 - 3t_{10} - p^2 t_{20})$$

$$+ p^{\lambda} \not{k} (p^2 t_4 - t_7 - 3t_{11} + t_{17} + \text{pku } t_{20} + t_{28})$$

$$+ p^{\lambda} i\sigma^{\alpha\beta} k_{\alpha} p_{\beta} (t_3 + t_8 + 4t_{12} - t_{18})$$

$$+ \gamma^{\lambda} (p^2 t_6 + \text{pku } t_7 - 3t_{13} + \text{pku } t_{22} + k^2 t_{23} + t_{25})$$

$$+ i\sigma^{\lambda\alpha} p_{\alpha} (t_5 - \text{pku } t_8 + 4t_{14} - k^2 t_{24} + t_{26})$$

$$+ i\sigma^{\lambda\alpha} k_{\alpha} (p^2 t_8 + 4t_{15} + t_{21} \text{ pku } t_{24} - t_{27})$$

$$+ \varepsilon^{\lambda\alpha\beta\phi} \gamma^5 \gamma_{\phi} k_{\alpha} p_{\beta} (t_7 - 3t_{16} + t_{22} + t_{28}) \tag{5-26}$$

These seven equations complete the work necessary to execute a check on the correctness of the right-hand side as it was described in Chapter IV.

CHAPTER VI

THE MASS SHELL SOLUTION

6-1 An Approximate Solution

In Chapter IV the transverse part of the vertex equation was decomposed into a set of eight linked differential equations out of the original integral expression for the matrix function. The eight differential equations established relations for the eight unknown scalar functions, F, G_0, G_1, G_2, H_0, H_1, H_2 and I. These scalar functions exactly describe the transverse vertex for a given photon momentum-squared, k^2.

$$\Gamma^{\tilde{\lambda}}(\bar{p}+\bar{k},\bar{p}) = \frac{p^{\tilde{\lambda}}}{p} F(p^2,u) + p^{\tilde{\lambda}}[\not{p},\not{k}] \frac{H_0}{kp^2} (p^2,u)$$

$$+ [\gamma^{\tilde{\lambda}},\not{p}] \frac{H_1}{p} (p^2,u) + [\gamma^{\tilde{\lambda}},\not{k}] \frac{H_2}{k} (p^2,u) + \gamma^{\tilde{\lambda}} G_0(p^2,u)$$

$$+ p^{\tilde{\lambda}}\not{p} \frac{G_1}{p^2} (p^2,u) + p^{\tilde{\lambda}}\not{k} \frac{G_2}{kp} (p^2,u) + \varepsilon^{\tilde{\lambda}\alpha\nu\phi}\gamma^5\gamma_\phi k_\alpha p_\nu \frac{I}{kp} (p^2,u). \quad (6\text{-}1)$$

The circumflex signifies the transverse component. The scalar functions are shown as functions of the electron momentum-squared, p^2, and the relative angle function, u, where $u = \frac{\bar{p}\cdot\bar{k}}{(|p^2||k^2|)^{\frac{1}{2}}}$. The scalar functions are parametrized by the value of k^2. The eight equations were checked by a process described in Chapter V. The eight linked equations which are third order in derivatives including all nine possible mixed

derivatives with respect to p^2 and u, were complicated to write down, difficult to check and promised to be significantly more difficult to solve. Therefore it was important to start with a good guess of the correct solution. From the experience of solving the electron equation, where it was learned that the mass shell solution dominated a wide region around the mass shell, it was hoped that the vertex equation would similarly be a slowly changing function. If this were so, then a solution to the eight differential equations which worked near the mass shell might be extended into the asymptotic region by simple variations of the eight scalar functions.

In 1981 H. S. Green communicated an approximate solution to the second order tensor equation,

$$\nabla^2\Gamma^\lambda = \varepsilon[\tfrac{1}{2}\gamma_\nu F^{\nu\lambda} + \partial_\nu \not{\nabla}^{-1} F^{\nu\lambda}]. \tag{6-2}$$

The third order tensor equation, Eq. (4-16), is obtained from this equation by the operation of $\not{\nabla}$. The tensor functions Green found to be approximate solutions to Eq. (6-2) were used to identify the approximate form of the eight scalar functions. The approximations employed were commensurate with the known behavior of the vertex and the electron propagator near the mass shell.

In Eq. (6-2) the tensor

$$F^{\nu\lambda} = S(\bar{p}_1)\frac{\partial}{\partial p_\nu}[\Gamma^\lambda(\bar{p}_1,\bar{p}_2)S(p_2)]S^{-1}(p_2) \tag{6-3}$$

was approximated by $\Delta^\lambda\gamma^\nu$, where

$$\Delta^\lambda = S(\bar{p}_1)\Gamma^\lambda(p_1,p_2)S(\bar{p}_2). \tag{6-4}$$

This amounts to neglecting

$$S(\bar{p}_1)[\frac{\partial}{\partial p_\nu} \Gamma^\lambda(\bar{p}_1\bar{p}_2) - \Gamma^\lambda(\bar{p}_1\bar{p}_2)S(p_2)(p_2)(\frac{\partial}{\partial p_\nu} S^{-1}(\bar{p}_2) - \gamma^\nu)]. \tag{6-5}$$

This term approaches zero as Γ^λ approaches its mass shell limit of γ^λ. Near the mass shell the appropriate form of the second order vertex equation is given by

$$\nabla^2\Gamma^\lambda = -\,\varepsilon[\tfrac{1}{2}\,\gamma_\nu\Delta^\lambda\gamma^\nu + \not{\nabla}^{-1}\Delta^\lambda\,\overleftarrow{\not{\nabla}}]. \tag{6-6}$$

This is a far simpler equation than (6-2).

This equation can be decomposed into a set of four linearly independent matrix equations by the following device. Take the product of Eq. (6-6) with 1, γ_μ, $\gamma_{\mu\nu}$ and $\gamma_{\mu\nu\rho}$, then take the trace of each product. To facilitate this the following definitions were made:

$$C^\lambda = \frac{1}{4}\,\mathrm{tr}\,[\Gamma^\lambda] \tag{6-7a}$$

$$C^\lambda_\mu = \frac{1}{4}\,\mathrm{tr}\,[\Gamma^\lambda\gamma_\mu] \tag{6-7b}$$

$$C^\lambda_{\mu\nu} = \frac{1}{4}\,\mathrm{tr}\,[\Gamma^\lambda\gamma_{\mu\nu}] \tag{6-7c}$$

$$C^\lambda_{\mu\nu\rho} = \frac{1}{4}\,\mathrm{tr}\,[\Gamma^\lambda\gamma_{\mu\nu\rho}] \tag{6-7d}$$

$$D^\lambda = \frac{1}{4}\,\mathrm{tr}\,[\Delta^\lambda] \tag{6-7e}$$

$$D^\lambda_\mu = \frac{1}{4}\,\mathrm{tr}\,[\Delta^\lambda\gamma_\mu] \tag{6-7f}$$

$$D^\lambda_{\mu\nu} = \frac{1}{4}\,\mathrm{tr}\,[\Delta^\lambda\gamma_{\mu\nu}] \tag{6-7g}$$

$$D^\lambda_{\mu\nu\rho} = \frac{1}{4}\,\mathrm{tr}\,[\Delta^\lambda\gamma_{\mu\nu\rho}] \tag{6-7h}$$

where $$\gamma_{\mu\nu} = \frac{1}{2}[\gamma_\lambda, \gamma_\mu]$$

and $$\gamma_{\mu\nu\rho} = \frac{1}{2}\{\gamma_\lambda, \gamma_{\mu\nu}\}. \tag{6-8}$$

Now it can be seen that the set of four equations below is equivalent to Eq. (6-6).

$$\nabla^2 C^\lambda = -\,3\varepsilon D^\lambda \tag{6-9a}$$

$$\nabla^2 C^\lambda_\mu = 2\varepsilon D^\lambda_\mu - 2\varepsilon\, \partial_\mu \nabla^2 \partial^\nu D^\lambda_\nu \tag{6-9b}$$

$$\nabla^2 C_{\mu\nu} = -\varepsilon D^\lambda_{\mu\nu} - 2\varepsilon\, \partial_\mu \nabla^{-2} \partial^\rho D^\lambda_{\nu\rho} + 2\varepsilon\, \partial_\nu \nabla^{-2} \partial^\rho D^\lambda_{\mu\rho} \tag{6-9c}$$

$$\nabla^2 C^\lambda_{\mu\nu\rho} = -\,2\varepsilon\, \partial_\mu \nabla^{-2} \partial^\sigma D^\lambda_{\nu\rho\sigma} - 2\varepsilon\, \partial_\nu \nabla^{-2} \partial^\sigma D^\lambda_{\rho\mu\sigma}$$

$$- 2\varepsilon\, \partial_\rho \nabla^{-2} \partial^\sigma D^\lambda_{\mu\nu\sigma}. \tag{6-9d}$$

The D^λ, D^λ_μ, $D^\lambda_{\mu\nu}$ and $D^\lambda_{\mu\nu\rho}$ tensors are evaluated by using the definition of Δ^λ, Eq. (6-4) in Eqs. (6-7e) through (6-7h).

$$D^\lambda = \frac{1}{4}\,\mathrm{tr}\,[\Gamma^\lambda(\not{p}_2 + A_2)(\not{p}_1 + A_1)]/D_{12}$$

$$= [(p_1 \cdot p_2 + A_1A_2)C^\lambda + (A_1p_2^\mu + A_2p_1^\mu)C^\lambda_\mu - p_1^\mu p_2^\nu C^\lambda_{\mu\nu}]/D_{12} \tag{6-10a}$$

$$D^\lambda_\mu = \frac{1}{4}\,\mathrm{tr}\,[\Gamma^\lambda(\not{p}_2 + A_2)\gamma_\mu(\not{p}_1 + A_1)]/D_{12}$$

$$= [(A_2p_{1\mu} + A_1p_{2\mu})C^\lambda + (A_1A_2 - \bar{p}_1 \cdot \bar{p}_2)C^\lambda_\mu + (p_{1\mu}p_2^\nu + p_{2\mu}p_1^\nu)C^\lambda_\nu$$

$$+ (A_2p_1^\nu - A_2p_1^\nu)C^\lambda_{\mu\nu} + (p_1^\nu p_2^\rho C^\lambda_{\mu\nu\rho})]/D_{12} \tag{6-10b}$$

$$D^\lambda_{\mu\nu} = \frac{1}{4}\,\mathrm{tr}\,[\Gamma^\lambda(\not{p}_2 + A_2)\gamma_{\mu\nu}(\not{p}_1 + A_1)]/D_{12}$$

$$= [-(p_{1\mu}p_{2\nu} - p_{2\mu}p_{1\nu})C^\lambda + (A_2p_{1\nu} - A_1p_{2\nu})C^\lambda_\mu - (A_2p_{1\mu} - A_1p_{2\mu})C^\lambda_\nu + (p_1 \cdot p_2 + A_1A_2)C^\lambda_{\mu\nu}$$

$$- (p_{1\nu}p_2^\rho + p_{2\nu}p_1^\rho)C^\lambda_{\mu\rho} + (p_{1\mu}p_2^\rho + p_{2\mu}p_1^\rho)C^\lambda_{\nu\rho} + (A_2p_1^\rho + A_1p_2^\rho)C^\lambda_{\mu\nu\rho}]/D_{12} \tag{6-10c}$$

$$D^\lambda_{\mu\nu\rho} = \frac{1}{4}\, \mathrm{tr}\, [\Gamma^\lambda(\not{p}_2+A_2)\gamma_{\mu\nu\rho}(\not{p}_1+A_1)]/D_{12}$$

$$= [(p_{1\mu}p_{2\nu}-p_{2\mu}p_{1\nu})C^\lambda_\rho + (p_{1\nu}p_{2\rho}-p_{2\nu}p_{1\rho})C^\lambda_\mu + (p_{1\rho}p_{2\mu}-p_{2\rho}p_{1\mu})C^\lambda_\nu$$

$$+ (A_2p_{1\mu}+A_1p_{2\mu})C^\lambda_{\nu\rho} + (A_2p_{1\nu}+A_1p_{2\nu})C^\lambda_{\rho\mu} + (A_2p_{1\rho}+A_1p_{2\rho})C^\lambda_{\mu\nu}$$

$$+ (A_1A_2-p_1\cdot p_2)C^\lambda_{\mu\nu\rho} + (p_{1\mu}p_2^\sigma+p_{2\mu}p_1^\sigma)C^\lambda_{\mu\rho\sigma} + (p_{1\nu}p_2^\sigma+p_{2\nu}p_1^\sigma)C^\lambda_{\rho\mu\sigma}$$

$$+ (p_{1\rho}p_2^\sigma+p_{2\rho}p_1^\sigma)C^\lambda_{\mu\nu\sigma}]/D_{12}. \tag{6-10d}$$

These definitions are somewhat intimidating in appearance but the full effect of assuming the vertex function will be dominated by γ^λ in the region of the mass shell has not yet been fully explored. If Γ^λ is well expressed by γ^λ for some range of values of p^2 and u, then this means C^λ_μ is well expressed by δ^λ_μ † and the other tensors C^λ, $C^\lambda_{\mu\nu}$ and $C^\lambda_{\mu\nu\rho}$ have negligible effect relative to C^λ_μ. Furthermore, the electron propagator has the simple form $Z_2/\not{p}-m$. Under these assumptions the D^λ, D^λ_μ, $D^\lambda_{\mu\nu}$ and $D^\lambda_{\mu\nu\rho}$ tensors take on the greatly simplified appearance of

$$D^\lambda = \frac{2mp^\lambda}{D_{12}} \tag{6-11a}$$

$$D^\lambda_\mu = [(m^2-\bar{p}_1\cdot\bar{p}_2)\delta^\lambda_\mu + (p_{1\mu}+p_{2\mu})p^\lambda]/D_{12} \tag{6-11b}$$

$$D^\lambda_{\mu\nu} = m[[(p_{1\nu}-p_{2\nu})\delta^\lambda_\mu - (p_{1\mu}-p_{2\mu})\delta^\lambda_\nu]/D_{12} \tag{6-11c}$$

$$D^\lambda_{\mu\nu\rho} = [(p_{1\nu}p_{2\nu}-p_{2\mu}p_{1\nu})\delta^\lambda_\rho + (p_{1\nu}p_{2\rho}-p_{2\nu}p_{1\rho})\delta^\lambda_\mu$$

$$+ (p_{1\rho}p_{2\mu}-p_{2\rho}p_{1\mu})\delta^\lambda_\nu]/D_{12} \tag{6-11d}$$

where $\quad D_{12} = (p_1^2-m^2)(p_2^2-m^2)$.

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† δ^λ_μ is defined in Appendix A.

Green's work using these definitions of the D^{λ} type tensors to solve the four equations, (6-9a) through (6-9d), for the four tensors C^{λ}, C^{λ}_{μ}, $C^{\lambda}_{\mu\nu}$ and $C^{\lambda}_{\mu\nu\rho}$, is reproduced in Appendix B. In the next three sections his expressions for C^{λ}, C^{λ}_{μ} , $C^{\lambda}_{\mu\nu}$ and $C^{\lambda}_{\mu\nu\rho}$ are used to determine the near-the-mass-shell forms of the eight scalar functions F, G_0, G_1, G_2, H_0, H_1 H_2 and I.

6-2 The F and I Functions

In Appendix B, Eq (B-32) gives the expression for the $C^{\tilde{\lambda}}$ tensor

$$C^{\tilde{\lambda}} = \frac{3}{4}\varepsilon\, mp^{\tilde{\lambda}} \int_{-1}^{1} \frac{d_{\beta}}{x_{\beta}} [\frac{u_{\beta}}{x_{\beta}} \ln\,(1 - \frac{x_{\beta}}{u_{\beta}}) + 1] \tag{6-12}$$

where

$$x_{\beta} = [\bar{p} \;\; + (\beta+1)\,\frac{\bar{k}}{2}]^{2} \text{ and } u_{\beta} = m^{2} - \frac{1}{4}(1-\beta^{2})k^{2}. \tag{6-13}$$

Furthermore, from the definition of $C^{\tilde{\lambda}}$, Eq. (6-7a), the scalar function F is determined.

$$C^{\tilde{\lambda}} = \frac{1}{4}\,\mathrm{tr}\,(\Gamma^{\tilde{\lambda}}) = p^{\lambda}\,\frac{F}{p}. \tag{6-14}$$

Combining Eqs. (6-12) and (6-14) the scalar function is identified in terms of an integration over beta.

$$F = \frac{3}{4}\varepsilon\, mp \int_{-1}^{1} \frac{d_{\beta}}{x_{\beta}} [\frac{u_{\beta}}{x_{\beta}} \ln(1-\frac{x_{\beta}}{u_{\beta}}) + 1]. \tag{6-15}$$

Obtaining the I function is slightly more involved but in the end it will be seen that the expression for I is simply related to the expression for F. The defining equation for $C^{\tilde{\lambda}}_{\mu\nu\rho}$ is given in Appendix B, Eq. (B-59) and it states,

$$\nabla^2(p_1^\mu p_2^\nu \partial^\rho C^{\tilde{\lambda}}_{\mu\nu\rho}) = \varepsilon\, k^2 \partial^{\tilde{\lambda}} \int_{-1}^{1} d\beta \frac{1}{(x_\beta - u_\beta)} . \tag{6-16}$$

Using the fact that when

$$\Phi = \frac{1}{4}\left(1 - \frac{u_\beta}{x_\beta}\right) \ln \left(1 - \frac{x_\beta}{u_\beta}\right) \tag{6-17}$$

then

$$\nabla^2 \Phi = \frac{1}{x_\beta - u_\beta} \tag{6-18}$$

so that Eq. (B-16) can be re-expressed as

$$\nabla^2(p_1^\mu p_2^\nu \partial^\rho C^{\tilde{\lambda}}_{\mu\nu\rho}) = \varepsilon\, k^2 \partial^{\tilde{\lambda}} \nabla^2 \int \Phi \; d\beta. \tag{6-19}$$

It follows that

$$p_1^\mu p_2^\nu \partial^\rho C^{\tilde{\lambda}}_{\mu\nu\rho} = \varepsilon\, k^2 \partial^{\tilde{\lambda}} \int \Phi \; d\beta. \tag{6-20}$$

From the definition of $C^{\lambda}_{\mu\nu\rho}$ the scalar function I is defined.

$$C^{\tilde{\lambda}}_{\mu\nu\rho} = \frac{1}{4} \operatorname{tr} [\Gamma^{\tilde{\lambda}} \gamma_{\mu\nu\rho}] = \{\delta^{\tilde{\lambda}}_{\mu} [k_\rho p_\nu - k_\nu p_\rho]$$
$$+ \delta^{\tilde{\lambda}}_{\nu}[k_\mu p_\rho - k_\rho p_\mu] + \delta^{\tilde{\lambda}}_{\rho}[k_\nu p_\mu - k_\mu p_\nu]\} \frac{I}{kp} . \tag{6-21}$$

From the above it can be shown that it follows that

$$p_1^\mu p_2^\nu \partial^\rho C^{\tilde{\lambda}}_{\mu\nu\rho} = 2 \frac{k}{p} p^{\tilde{\lambda}} I . \tag{6-22}$$

From Eq. (6-17) it can be shown that

$$\partial^{\tilde{\lambda}} \int \Phi \; d\beta = \int 2p^{\tilde{\lambda}} \left\{ \frac{u_\beta^2}{x_\beta^2} \ln \left(1 - \frac{x_\beta}{u_\beta}\right) + \frac{1}{x_\beta} \right\} d\beta. \tag{6-23}$$

By substituting Eqs. (6-22) and (6-23) into Eq. (6-20) the function I is identified,

$$I = \frac{\varepsilon}{4} pk \int_{-1}^{1} \left[\frac{u_\beta}{x_\beta^2} \ln\left(1 - \frac{x_\beta}{u_\beta}\right) + \frac{1}{x_\beta}\right] d\beta. \tag{6-24}$$

It is now apparent that a simple relationship has emerged. That is,

$$I = \frac{k}{3m} F. \tag{6-25}$$

6-3 The G_0, G_1, and G_2 Functions

The C_μ^λ tensor is defined in the appendix by the following statement.

$$C_\mu^* = C^* \delta_\mu^\lambda - \partial^\lambda C_\mu^* \tag{B-48}$$

where

$$C^* = \frac{\varepsilon}{4} \int_{-1}^{1} \left[1 - \frac{u_\beta}{x_\beta} - \frac{k^2}{2x_\beta}\right] \ln\left(1 - \frac{x_\beta}{u_\beta}\right) d\beta$$

$$- \frac{\varepsilon}{4} \left\{\left(1 - \frac{1}{p_1^2}\right) \ln\left(1 - p_1^2\right) + \left(1 - \frac{1}{p_2^2}\right) \ln\left(1 - p_2^2\right)\right\} \tag{B-40}$$

and where

$$C_\mu^* = \frac{\varepsilon}{4} \tilde{p}_\mu \int_{-1}^{1} (m^2 - u_\beta) \frac{1}{x_\beta} \left[\left(\frac{u_\beta}{x_\beta} - 1\right) \ln\left(1 - \frac{x_\beta}{u_\beta}\right) + 1\right] d\beta$$

$$- \frac{\varepsilon}{4} k_\mu \left(\frac{p \cdot k}{k^2} + \frac{1}{2}\right) \int_{-1}^{1} (m^2 - u_\beta) \frac{1}{x_\beta} \ln\left(1 - \frac{x_\beta}{u_\beta}\right) d\beta. \tag{B-47}$$

For greater ease in manipulating these quantities, the following definitions are made:

$$\int I_2 d\beta = \int_{-1}^{1} \left(1 - \frac{u_\beta}{x_\beta} - \frac{k^2}{2x_\beta}\right) \ln\left(1 - \frac{x_\beta}{u_\beta}\right) d\beta \tag{6-26}$$

$$\int I_3 d\beta = \int_{-1}^{1} (1 - u_\beta) \frac{1}{x_\beta} [(\frac{u_\beta}{x_\beta} - 1) \ln (1 - \frac{x_\beta}{u_\beta}) + 1] d\beta \tag{6-27}$$

$$\int I_4 d\beta = \int_{-1}^{1} (1 - u_\beta) \frac{1}{x_\beta} \ln (1 - \frac{x_\beta}{u_\beta}) \tag{6-28}$$

$$CK = (1 - \frac{1}{p_1^2}) \ln (1 - p_1^2) + (1 - \frac{1}{p_2^2}) \ln (1 - p_2^2). \tag{6-29}$$

Then

$$\partial^\lambda C_\mu^* = \frac{1}{4} \varepsilon \delta_\mu^\lambda \int I_3 d\beta + \frac{1}{4} \varepsilon \tilde{p}_\mu \frac{\partial}{\partial p_\lambda} x_\beta \int I_3^* d\beta$$

$$- \frac{\varepsilon}{4} k_\mu \frac{\partial}{\partial p^\lambda} [p \frac{k_\mu}{k^2} + \frac{1}{2}] I_4 d\beta - \frac{\varepsilon}{4} k_\mu [p \frac{k_\mu}{k^2} + \frac{1}{2}] \frac{\partial x_\beta}{\partial p_\lambda} \int I_4^x d\beta \tag{6-30}$$

where

$$\frac{\partial x_\beta}{\partial p^\lambda} = 2p^\lambda; \; I^x = \frac{\partial I}{\partial x_\beta} ; \; \frac{\partial pk_\mu}{\partial p^\lambda} = 0; \text{ and } \tilde{p}_\mu = p_\mu - k_\mu \frac{pk_\mu}{k^2} .$$

Then

$$\partial^\lambda C_\mu^* = \frac{1}{4} \varepsilon \delta_\mu^\lambda \int I_3 d\beta + \frac{\varepsilon}{2} p_\mu p^\lambda \int I_3^x d\beta - \frac{\varepsilon}{2} k_\mu p^\lambda \frac{pk_\mu}{k^2} \int I_3^x d\beta$$

$$- \frac{\varepsilon}{2} k_\mu [\frac{pk_\mu}{k^2} + \frac{1}{2}] p^\lambda \int I_4^x d\beta \tag{6-31}$$

and

$$C^* = \frac{\varepsilon}{4} \int I_2 d\beta - \frac{\varepsilon}{4} CK \tag{6-32}$$

so that

$$C_\mu^\lambda = C^* \delta_\mu^\lambda - \partial^\lambda C_\mu^*$$

$$= \delta_\mu^\lambda [\frac{\varepsilon}{4} \int I_2 d\beta - \frac{\varepsilon}{4} CK - \frac{\varepsilon}{4} \int I_3 d\beta]$$

$$+ p^\lambda p_\mu[- \tfrac{1}{2} \varepsilon \int I_3^x d\beta$$

$$+ p^\lambda k_\mu[\tfrac{1}{4}\varepsilon \left(2 \frac{pk_\mu}{k^2} + 1\right)\left(\int I_3^x d\beta + I_4^x d\beta\right) - \tfrac{1}{4} \varepsilon \int I_3^x d\beta]. \quad (6\text{-}33)$$

From the definition of C_μ^λ, Eq. (6-7b), the identification of G_o, G_1, and G_2 can be made.

$$C_\mu^\lambda = \frac{1}{4} \text{tr}\, [\Gamma^\lambda \gamma_\mu]$$

$$= \delta_\mu^\lambda G_o + p^\lambda p_\mu \frac{G_1}{p_2} + p^\lambda k_\mu \frac{G_2}{pk} . \quad (6\text{-}34)$$

From a comparison of Eqs. (6-33) and (6-34) the final expressions for the three G functions are formed.

$$G_o = \frac{\varepsilon}{4} \int I_2 d\beta - \frac{\varepsilon}{4} CK - \frac{\varepsilon}{4} \int I_3 d\beta \quad (6\text{-}35)$$

$$G_1 = - \frac{p^2 \varepsilon}{2} \int I_3^x d\beta \quad (6\text{-}36)$$

$$G_2 = \frac{\varepsilon}{4} (2p^2 u + pk)\left(\int I_3^x d\beta + \int I_4^x d\beta\right)$$

$$- \frac{\varepsilon}{4} \int I_3^x \, pk d\beta. \quad (6\text{-}37)$$

6-4 The H_o, H_1, H_2 Functions

The defining equation for $C_{\mu\nu}^\lambda$ is shown in the appendix to be

$$C_{\mu\nu}^\lambda = \frac{\varepsilon m}{16} [\delta_\mu^\lambda k_\alpha \partial^\alpha - k_\mu \partial^\lambda)\partial_\nu - (\delta_\nu^\lambda k_\alpha \partial^\alpha - k_\mu \partial^\lambda)\partial_\mu] \int I_5 d\beta$$

$$+ \frac{\varepsilon m}{8} (\delta_\mu^\lambda k_\nu - \delta_\nu^\lambda k_\mu) \int I_6 d\beta \quad (B\text{-}55)$$

where

$$\int I_5 d\beta = \int_{-1}^{1} [L_2(\frac{x_\beta}{u_\beta}) + (1 - \frac{x_\beta}{u_\beta}) \ln (1 - \frac{x_\beta}{u_\beta})]d\beta \quad (6\text{-}38)$$

$$\int I_6 d\beta = \int_{-1}^{1} [\frac{1}{x\beta} \ln (1 - \frac{x_\beta}{u_\beta})] d\beta \tag{6-39}$$

$$L_2(z) = - \int_o^z \ln(1-z) \frac{dz}{z}. \tag{6-40}$$

In order to perform the implied operations first notice that

$$\partial_\alpha I(x_\beta) = \frac{\partial x_\beta}{\partial p_\alpha} \frac{\partial I}{\partial x_\beta} = (2p_\alpha + (\beta+1)k_\alpha) I^x \tag{6-41}$$

where

$$I^x = \frac{\partial I}{\partial x_\beta} \tag{6-42}$$

and

$$\partial^\alpha \partial_\eta I = 2\delta^\alpha_\eta I^x + k_\eta k^\alpha (\beta+1)^2 I^{xx} + p^\alpha k_\nu 2(\beta+1) I^{xx}$$

$$+ 2p_\eta k^\alpha (\beta+1) I^{xx} + 4p_\eta p^\alpha I^{xx}. \tag{6-43}$$

Completing the operations implied in Eq. (B-55) yields a new expression.

$$\begin{aligned} C^\lambda_{\mu\nu} = \frac{\epsilon m}{8} \{ &(p^\lambda p_\mu k_\nu - p^\lambda p_\nu k_\mu)\, 2 \int I_5^{xx}\, d\beta \\ &+ (\delta^\lambda_\mu k_\nu - \delta^\lambda_\nu k_\mu)[2 \int I_5^x d\beta + 2k^2 \int I_5^{xx} \beta^2 d\beta \\ &+ (2pku + 2k^2) \int I_5^{xx}\, \beta d\beta + 2pku \int I_5^{xx} d\beta + \int I_6 d\beta] \\ &+ (\delta^\lambda_\mu p_\nu - \delta^\lambda_\nu p_\mu)[4k^2 \int I_5^{xx} \beta d\beta + 4pku \int I_6^{xx} d\beta]\}. \end{aligned} \tag{6-44}$$

The relation between the tensor function $C^\lambda_{\mu\nu}$ and the three scalar H functions is given by the definition

$$C^\lambda_{\mu\nu} = \frac{1}{4} \mathrm{tr}\, [\Gamma^\lambda \gamma_{\mu\nu}]. \tag{6-7c}$$

This can be expanded to reveal the relation between $C^{\lambda}_{\mu\nu}$ and the three H functions.

$$C^{\lambda}_{\mu\nu} = (p^{\lambda}p_{\nu}k_{\mu}-p^{\lambda}p_{\mu}k_{\nu})\,\frac{2H_o}{kp^2}$$

$$+ (\delta^{\lambda}_{\nu}k_{\mu}-\delta^{\lambda}_{\mu}k_{\nu})\,\frac{2H_1}{k}$$

$$+ (\delta^{\lambda}_{\nu}p_{\mu}-\delta^{\lambda}_{\mu}p_{\nu})\,\frac{2H_2}{p}\ . \tag{6-46}$$

By comparison of Eqs. (6-44) and (6-46) the three H functions are now expressible in the following way,

$$H_o = -\frac{\varepsilon}{8}\,kp^2\int I^{xx}_5 d\beta \tag{6-47}$$

$$H_1 = \frac{\varepsilon m}{16}\,[-\,pk^2\int I^{xx}_5\beta d\beta - 2(p^2ku+\frac{k^2p}{2})\int I^{xx}_5 d\beta] \tag{6-48}$$

$$H_2 = \frac{\varepsilon mk}{8}\,[\int I^{x}_5\,d\beta + \frac{1}{2}\int I_6 d\beta]$$

$$+ \frac{\varepsilon mk}{8}\,[(\frac{pku}{2}+\frac{k^2}{4})\int I^{xx}_5 d\beta]$$

$$+ \frac{\varepsilon mk^3}{32}\int I^{xx}_5\beta^2 d\beta + \frac{\varepsilon mk}{16}\,(pku+\frac{k^2}{2})\int I^{xx}_5\beta d\beta. \tag{6-49}$$

6-5 Summary of the Mass Shell Solution

All of the preceding description of the eight scalar functions, F, G_o, G_1, G_2, H_o, H_1, H_2 and I has established the transverse part of the vertex function in any region which is typified by the satisfaction of two constraints. One constraint is that the electron propagator be well represented by

$$\frac{1}{\not p - m}\ ,$$

where m is the experimental mass of the electron. This means that the function $A(p_1^2)$ or $A(p_2^2)$ must be essentially constant and equal to the experimental mass. The second constraint is that $\Gamma^{\tilde{\lambda}}$ must be dominated by the contribution from $\tilde{\gamma}^{\lambda}$ so that the transverse vertex is expressible in a series expansion,

$$\Gamma^{\tilde{\lambda}} = \gamma^{\tilde{\lambda}} + \frac{\alpha}{\pi} L_1^{\tilde{\lambda}} + (\frac{\alpha}{\pi})^2 L_2^{\tilde{\lambda}} + (\frac{\alpha}{\pi})^3 L_3^{\tilde{\lambda}} + \ldots \tag{6-50}$$

where α is the fine structure constant. If it is possible to assume that the first term in the expansion dominates and if the electron propagator is on the mass shell then it is expected that the functions F through I will satisfy the eight differential equations which have been written into the Main Program. These eight function solutions have been collected together for ease of reference in Table 6-1.

By using $\gamma^{\tilde{\lambda}}$ as the first order contribution to Γ^{λ} a solution was found to the vertex equation up to second order. In effect the solution which was determined represents an identification of the $L_1^{\tilde{\lambda}}$ term in Eq. (6-50). If this new improved version of $\Gamma^{\tilde{\lambda}} = \gamma^{\tilde{\lambda}} + \frac{\alpha}{\pi} L_1^{\tilde{\lambda}}$ were put into the vertex equation the vertex could be determined up to third order. There is a practical limit to how far this process should be carried toward self consistency. The coupling constant, $\frac{\alpha}{\pi}$, is a very small number, 2.32×10^{-3}. Where it is true that $\Gamma^{\tilde{\lambda}}$ is dominated by $\gamma^{\tilde{\lambda}}$ the third order corrections would make little difference to the solution. An even more important consideration; the determination of the vertex function is only one step in a larger iterative procedure which seeks to re-evaluate the electron and photon propagators to self consistency.

TABLE 6-1. THE MASS SHELL FUNCTIONS

$$F = \frac{3}{4}\, \varepsilon m p \int I_1 d\beta$$

$$I = \frac{\varepsilon}{4}\, pk \int I_1\, d\beta$$

$$G_o = 1 + \frac{\varepsilon}{4} \int I_2 d\beta - \frac{\varepsilon}{4}\, CK - \frac{\varepsilon}{4} \int I_3 d\beta$$

$$G_1 = -\frac{\varepsilon}{2}\, p^2 \int I_3^x\, d\beta$$

$$G_2 = \frac{\varepsilon}{4}\, pk \left(\frac{2pku}{k^2} + 1\right)\left[\int I_3^x d\beta + \int I_4^x d\ \right] - \frac{\varepsilon}{4}\, pk \int\ I^x d_\beta$$

$$H_o = -\frac{\varepsilon}{8}\, kp^2 m \int I_5^{xx} d\beta$$

$$H_1 = \frac{\varepsilon m}{16}\left[-\ pk^2 \int I_5^{xx} \beta d\beta - 2\left(p^2 ku + \frac{k^2 p}{2}\right) \int I_5^{xx} d\beta\right]$$

$$H_2 = \frac{\varepsilon m k}{8}\left[\int I_5^x d\beta + \frac{1}{2}\int I_6 d\beta\right] + \frac{\varepsilon m k}{8}\left(\frac{pku}{2} + \frac{k^2}{4}\right)\left[\int I_5^{xx} d\beta + \int I_5^{xx} \beta d\beta\right]$$
$$+ \frac{\varepsilon m k^3}{32} \int I_5^{xx}\ \beta^2 d\beta.$$

$$I_1 = \frac{1}{x_\beta}\left[\frac{u_\beta}{x_\beta} \ln\left(1 - \frac{x_\beta}{u_\beta}\right) + 1\right]$$

$$I_2 = \left(1 - \frac{u_\beta}{x_\beta} - \frac{k^2}{2x_\beta}\right) \ln\left(1 - \frac{x_\beta}{u_\beta}\right)$$

$$I_3 = (m^2 - u_\beta)\, \frac{1}{x_\beta}\left[\left(\frac{u_\beta}{x_\beta} - 1\right) \ln\left(1 - \frac{x_\beta}{u_\beta}\right) + 1\right]$$

$$I_4 = (m^2 - u_\beta)\, \frac{1}{x_\beta} \ln\left(1 - \frac{x_\beta}{u_\beta}\right)$$

$$CK = \left(1 - \frac{m^2}{p_1^2}\right) \ln\left(1 - \frac{p_1^2}{m^2}\right) + \left(1 - \frac{m^2}{p_2^2}\right) \ln\left(1 - \frac{p_2^2}{m^2}\right)$$

TABLE 6-1 (Continued)

$$I_3^x = (m^2-u_\beta)[-\frac{2u_\beta}{x_\beta^2}\ln(1-\frac{x_\beta}{u_\beta}) + \frac{1}{x_\beta^2}\ln(1-\frac{x_\beta}{u_\beta}) - \frac{2}{x_\beta^2}]$$

$$I_4^x = (m^2-u_\beta)[-\frac{1}{x_\beta^2}\ln(1-\frac{x_\beta}{u_\beta}) - \frac{1}{u_\beta x_\beta}\frac{1}{(1-\frac{x_\beta}{u_\beta})}]$$

$$I_5 = L_2(\frac{x_\beta}{u_\beta}) + (1-\frac{u_\beta}{x_\beta})\ln(1-\frac{x_\beta}{u_\beta})$$

$$L_2(z) = -\int_0^z \ln(1-z)\frac{dz}{z}$$

$$I_6 = \frac{1}{x_\beta}\ln(1-\frac{x_\beta}{u_\beta})$$

$$I_5^x = \frac{1}{u_\beta}[-\frac{u_\beta}{x_\beta}\ln(1-\frac{x_\beta}{u_\beta}) + \frac{u_\beta^2}{x_\beta^2}\ln(1-\frac{x_\beta}{u_\beta}) + \frac{u_\beta}{x_\beta}]$$

$$I_5^{xx} = \frac{1}{u_\beta^2}[\frac{u_\beta^2}{x_\beta^2}\ln(1-\frac{x_\beta}{u_\beta}) + \frac{u_\beta}{x_\beta}\frac{1}{(1-\frac{x_\beta}{u_\beta})} - \frac{2u_\beta^3}{x_\beta^3}\ln(1-\frac{x_\beta}{u_\beta}) - \frac{u_\beta^2}{x_\beta^2}\frac{1}{(1-\frac{x_\beta}{u_\beta})} - \frac{u_\beta^2}{x_\beta^2}]$$

where

$$x_\beta = [\bar{p} + \frac{(\beta+1)}{2}\bar{k}]^2$$

$$u_\beta = m^2 - \frac{1}{4}(1-\beta^2)k^2.$$

$$\varepsilon = \frac{\alpha}{\pi}$$

α = fine structure constant.

This project involved an initial approximation for the photon propagator, from that a calculation of the electron propagator, from that a calculation of the vertex. Future plans for the project looked toward using the information gained by this work to calculate the photon propagator. From this new photon propagator a new electron propagator could be calculated. From these two a new vertex. Each cycle of calculation should contribute a finer resolution of the exact solutions. There would be no point in refining the vertex function much beyond the level to which the electron propagator is known. The functional form of the electron propagator is accurate to about 0.1% of the value of the function. Where γ^λ is dominant, corrections to order $(\frac{\alpha}{\pi})^2$ in $\tilde{\Gamma}^\lambda$ would be expected to amount to less than a 0.1% difference.

A direct consequence of solving the vertex equation only up to the $\frac{\alpha}{\pi}$ $L_1^{\tilde{\lambda}}$ term will be that the eighth equation, the coefficient of $\varepsilon^{\lambda\nu\mu\phi}\gamma^5\gamma_\phi k_\nu p_\mu$, will not be solved. It happens that the right-hand side of the eighth equation is second order in $\frac{\alpha}{4\pi}$. Therefore to this level of solution the eighth equation should equal zero on the right-hand side. For this reason equation eight will not be used as a criterion in assessing the correctness of the solution.

In the next chapter a test of the viability of the mass shell solution is made, and a description of the programming methods necessary to enact it is given.

CHAPTER VII

VERIFICATION OF THE MASS SHELL SOLUTION

7-1 The Mass Shell Program

Chapter VI established the functional forms of the eight scalar vertex functions. There is no single absolutely superior way to compile these functions which were summarized in Table 6-1. The Mass Shell Program--the Fortran Program--which computes the values of the functions at given points, assumed many strategies. The decision of how to compute the functions was influenced by concern for ease of assembly, the demand for machine time, and the inherent error in each method. Three basic categories of method were tested, and we became familiar with their merits. These three categories are described in section 7-3.

Once the mass shell solution was computed by one of these methods, it had to be interfaced with the Main Program by way of a matrix of numbers. The data matrix contains the value of each of the eight scalar functions and the nine possible partial derivatives of each function at one or more points in the infinite plane of the variables p^2 and u. Also included as data are the simpler electron propagator functions A and B (which were given in Eqs. (3-15) and (3-16)) and their derivatives. The Main Program uses the data matrix to evaluate the left- and right-hand sides (L.H.S. and R.H.S.) of each of the eight equations at each point provided. The Main Program evaluates the relative error in each equation at each point.

$$\text{Relative Error of Equation } i = \frac{(\text{L.H.S.} - \text{R.H.S.}) \text{ of Equation } i}{\text{L.H.S. of Equation } i} \quad (7\text{-}1)$$

$$\text{for } i = 1,2,\ldots,8.$$

The relative error is reported as a percentage error in the output. An optimal solution will have a minimal error over the entire plane defined by p^2 and u. The question of what constitutes an acceptable minimal error invites an analysis of what are the possible sources of error which contribute to the Main Program and the Mass Shell Program.

The relative error in the eight differential equations is the effect of a number of contributory causes. There is inherent error in the data matrix just because the Mass Shell Solution is an approximate solution to the eight differential equations. There is inherent error in the electron propagator functions A and B because they represent only approximate solutions to the electron equation. There are truncation errors introduced by the numerical procedures used to perform integrations and differentiations. Finally there is roundoff error--the inevitable outcome of any calculation which is carried out to a fixed finite number of figures. All of these sources of error had to be either maintained below a preset, tolerable level or, where they could not be controlled, they at least had to be understood well enough so that we could recognize when they were contributing to a significant loss of information. The next section will consider what were the sources of error in the Main Program and how these were controlled. The following section will discuss the three principal methods used to evaluate the vertex functions, F, G_0, G_1, G_2, H_0, H_1, H_2 and I, and how each method affected the overall level of uncertainty.

7-2 Contributions to Error in the Main Program

For the purpose of analyzing the kind of error that is being generated within the Main Program alone, we will assume for the time being that the data matrix of the eight scalar functions, their derivatives, the two scalar electron functions and their derivatives, could be supplied to the Main Program without error. If this could be done what would be the remaining sources of error and how would they propagate through the Main Program?

One of the earliest versions of the Main Program defined the partial derivatives of the twenty-eight $T^{\lambda\alpha}$ components by taking first-order differences. (This tensor appears in the R.H.S. and it was defined in Eqs. (4- 95) through (4-116).) This saved writing the analytic expressions for the partial derivatives of the twenty-eight components of $T^{\lambda\alpha}$ which, in turn, had to be expressed in terms of the partial derivatives of twenty-eight components of $F^{\lambda\nu}$ which, in turn, had to be expressed in terms of the partial derivatives of the twenty-eight components of $Z^{\lambda\nu}$ and $W^{\lambda\nu}$ which, in turn, were at last expressed in terms of the partial derivatives of the eight vertex functions and two electron functions provided by the data matrix. (See Figure 5-1 to be reminded of the hierarchy of tensors which define the R.H.S.) The numerical procedure for evaluating the partial derivatives of the twenty-eight components of $T^{\lambda\alpha}$ was quickly abandoned because the $T^{\lambda\alpha}$ components are very often extremely large and slowly changing, and roundoff error eliminated most useful details. It is a characteristic of the R.H.S. that, not only are the $T^{\lambda\alpha}$ components large, but that the R.H.S. is very much smaller

than its components. The R.H.S. is very sensitive to roundoff error because it involves the difference of many large and almost equal terms. In order to effectively calculate the R.H.S. analytic expressions of the partial derivatives of the components of the $T^{\lambda\alpha}$, $F^{\lambda\nu}$, $Z^{\lambda\nu}$ and $W^{\lambda\nu}$ tensors had to be supplied and the data matrix had to provide the functions and their derivatives to more than six figures.

Once the numerical means of calculating the derivatives of the components of $T^{\lambda\alpha}$ was discarded, the only remaining source of error in the Main Program is the accumulative effects of roundoff error. In hexadecimal based arithmetic the roundoff error for a single operation will be proportional to the 16^{-t+1}, where t is the number of digits in the mantissa when the number is expressed as a fraction times the base raised the power of the exponent. For a calculation of standard precision, the number of figures in the mantissa is only 8. This is not enough to provide an accurate evaluation of the R.H.S. It was necessary to increase this precision to 16 significant figures. This meant the roundoff error for a single operation was proportional to 16^{-15}, a very small number. However as the hundreds of thousands of single operations of multiplication and addition take place this error will grow systematically through the program. As mentioned before, the problem grows particularly severe on the R.H.S. of the vertex equation when the differences of large and nearly equal quantities are taken. It would be a huge task to draw a process graph to follow the propagation of the approximate error throughout the program. Even if this was done the projected error would be an upperbound with a large possible

deviation since the error would be assumed maximal at each individual step. As an alternative measure to estimate the intrinsic roundoff error, both the Main Program and the Mass Shell Program were converted to quadruple precision. The stability of the program results under the change from double precision to quadruple precision was taken to indicate that the data matrix was being supplied with numbers of sufficient accuracy that roundoff error was not a matter of principal concern. In this way the Main Program was established to be working satisfactorily. The really significant problem of error management lay in the design of the Mass Shell Program.

7-3 The Mass Shell Program

The Mass Shell Program takes the functional form of the F, G_o, G_1, G_2, H_o, H_1, H_2 and I functions and computes the value of the functions and all the derivatives at selected points. The vertex functions are parametrized by k^2 and are dependent on the variables p^2 and u, where

$$u = \frac{p_\alpha k^\alpha}{pk} = \frac{p_\alpha k^\alpha}{(p^2)^{\frac{1}{2}}(k^2)^{\frac{1}{2}}} . \qquad (7\text{-}2)$$

Due to the indefinite metric the domain of p^2 is from $-\infty$ to $+\infty$.

$$\begin{aligned} p^2 &= p_\mu p^\mu \\ &= g_{\mu\nu} p^\nu p^\mu . \end{aligned} \qquad (7\text{-}3)$$

Therefore both $(p^2)^{\frac{1}{2}}$ and $(k^2)^{\frac{1}{2}}$ can take on imaginary values. This leaves a dilemma in the interpretation of the meanings of the symbols p and k used in the definition of u and in the definitions of the functions in

Table 6-1. Initially this difficulty was circumvented, rather than resolved, by merely agreeing to consider only positive values of p^2 and k^2. In the following chapter designs are included to allow for the negative momenta and the full range of u.

Before agreeing to consider only the positive momentum squared, it was possible to tackle the real business of turning Table 6-1 into a reliable program which could provide a matrix of numbers representing the eight functions at a given selection of points in (p^2,μ,k^2) space. Also needed are the nine possible mixed derivatives of each vertex function. Let $\Gamma(p^2,\mu)$ represent any one of the eight vertex functions. Then

$$\not\partial\Gamma(p^2,\mu) = \gamma^\alpha \frac{\partial\Gamma}{\partial p^\alpha}$$

$$= \gamma^\alpha[2p_\alpha\Gamma' + \left(\frac{k_\alpha}{kp} - \frac{p_\alpha\mu}{p^2}\right)\Gamma^*]$$

$$= 2\not p\Gamma' + \frac{\not k\Gamma^*}{kp} - \frac{\not p\mu\Gamma^*}{p^2} \tag{7-4}$$

where

$$\Gamma' = \frac{\partial\Gamma}{\partial p^2} \qquad \text{and} \qquad \Gamma^* = \frac{\partial\Gamma}{\partial\mu}.$$

In order to obtain the left-hand side of the vertex equation, $\not\partial^3\Gamma^\lambda$, it is necessary to evaluate Γ', Γ'', Γ''', Γ^*, Γ^{**}, Γ^{***}, $\Gamma^{*'}$, $\Gamma^{**'}$ and $\Gamma^{*''}$ for each vertex function.

The object of the Mass Shell Program was to evaluate by some method the many types of integrals over beta which define the vertex function, and then by some other method obtain all of the above mentioned partial

derivatives. At the same time the Mass Shell Program was constrained to evaluate these functions and derivatives in a way which was conservative of machine time since many evaluations at different point were going to be made. It was also constrained to be very precise in its evaluations since it had been found that the R.H.S. of the vertex equation involved the subtraction of nearly equal quantities. This required that the data matrix be reliable to more than six significant figures. The evaluation of the beta integrals was a challenge to both of these constraints since the integrands become singular at those values of the integration variable where

$$\frac{x_\beta}{u_\beta} = 0 \qquad \text{or} \qquad \frac{x_\beta}{u_\beta} = 1. \tag{7-5}$$

Any integration method used has to be preceded by an identification of any such singular points so that a treatment could be prescribed.

A number of different approaches were made at forming the Mass Shell Program. They are divided into three main types. The first type chosen was to evaluate the functions with extreme precision and then take first, second and third order differences to form the derivatives. The second type wrote out the derivatives in terms of the analytic expressions derived from Table 6-1. Then the functions and the derivatives were all evaluated through integrations over a great number of beta integrals. The third type of approach was to perform analytic evaluations of the beta integrals to define the functions, then apply the method of taking differences to obtain the derivatives. Each method has its own special advantages and disadvantages which shall be further discussed. Familiarity

with the peculiarities of each approach led to a prescription for the most efficient acquisition of the data matrix.

Method 1

One way to determine all the scalar functions and their derivatives was to determine the functions to a minimum of 12 significant figures at a number of points and use these values to evaluate the derivatives by the method of taking differences. The main advantage of this method is that it performs the minimum number of different integrations over beta.

Two principal schemes were used to perform the integrations over beta, Gaussian quadrature and a modified trapezoidal rule. For the Gaussian quadrature subroutine, ninety-six points were used. This meant that if the integrand under the beta integral could be well approximated by a polynomial of the ninety sixth degree or less, there would be no truncation error involved in the use of the subroutine. Gaussian quadrature is typically fast and efficient. The subroutine used generally gave excellent results in a small fraction of time consumed by the trapezoidal rule subroutine. However, where the range of integration included a point where the integrand was singular, Gaussian quadrature is not expected to be reliable.

A monitor was needed to signal the occurrence of places where the integration needed to be performed in a different manner. This was done by writing out the analytic expressions for the first partial derivatives of each function in terms of the beta integrals and then evaluating the

first partial derivations by both the integrations over beta and by the first order differences of the functions. Where the two evaluations of the first order partial derivatives did not agree to at least eight significant figures, a warning message was automatically issued. This monitoring device was very useful as an alert. However it didn't solve the problem of the singular points. It only brought them to one's attention.

To deal with these special points in the integration over beta, an additional subroutine was added. This subroutine provided an alternative route to evaluate the beta integrals. It was a more time consuming path and therefore its use was flagged by the positive identification of a singular point. The problem integrals were then reexpressed so that the integrands no longer diverge. As an illustration, consider the following integral which is used in the definition of the F function,

$$\int_{-1}^{1} \frac{u_\beta}{x_\beta^2} \ln \left(1 - \frac{x_\beta}{u_\beta}\right) d\beta .$$

In the case that there is a β_o such that for $1 \leqslant \beta_o \leqslant 1$ there exists

$$\left.\frac{x_\beta}{u_\beta}\right]_{\beta=\beta_o} = \frac{x_{\beta o}}{u_{\beta_o}} = 1,$$

then $\ln \left(1 - \frac{x_{\beta_o}}{u_{\beta_o}}\right)$ will diverge. The integral can be reexpressed.

Let

$$Z = \frac{x_\beta}{u_\beta} ; \qquad Z_o = \frac{x_{\beta_o}}{u_{\beta o}} \qquad \text{and} \qquad \frac{\partial Z_o}{\partial \beta_o} = \left.\frac{\partial Z}{\partial \beta}\right]_{\beta=\beta_o} .$$

Then

$$\int_{-1}^{1} \frac{u_\beta}{x^2} \ln (1-Z)d = \int_{-1}^{1} \ln (1-Z)\left[\frac{u_\beta}{x_\beta^2} \frac{u_{\beta_0}}{x_{\beta_0}^2} \frac{\frac{\partial Z}{\partial \beta}}{\frac{\partial Z_0}{\partial \beta_0}}\right] d\beta$$

$$+ \frac{\frac{u_\beta}{x_{\beta_0}^2}}{\frac{\partial Z_0}{\partial \beta_0}} \int_{Z(\beta=-1)}^{Z(\beta=+1)} \ln (1-Z)dZ$$

$$= \int_{-1}^{1} \ln (1-Z)\left[\frac{u_\beta}{x_\beta^2} \frac{u_{\beta_0}}{x_{\beta_0}^2} \frac{\frac{\partial Z}{\partial \beta}}{\frac{\partial Z_0}{\partial \beta_0}}\right] d\beta$$

$$+ \frac{\frac{u_{\beta_0}}{x_{\beta_0}^2}}{\frac{\partial Z_0}{\partial \beta_0}} \Big[(Z-1) \ln (1-Z) + (1-Z) \Big]_{Z(\beta-1)}^{Z(\beta+1)} .$$

In this way, that part of the integral which the numerical technique evaluates with the greatest difficulty has been performed analytically.

Another way to handle the special points was to substitute a kind of trapezoidal rule for the Gaussian quadrature. The trapezoidal rule is in general less sensitive to the occurrence of isolated special points. An efficient modification of the trapezoidal rule, called Romberg integration,[25] was applied. The Romberg method is an example of a deferred approach to the limit. It applies the trapezoidal rule repeatedly. Each iteration alters the step size in a specific way such that the largest contribution to the truncation error at each step is cancelled by the following iteration.

Although the Romberg subroutine was slow and expensive compared to the Gaussian subroutine, it did provide a more certain evaluation of the integrals because the level of precision desired was preset by the tolerance. This routine was committed to grind away, subdividing the area under the integrand into small and smaller trapezoids, until the required self consistency was attained.

The calculation of the derivatives of the functions by the numerical method loses much of the precision which is attained by the Gaussian or Romberg integration subroutines. The nine partial derivatives of each of the eight vertex functions were calculated from the evaluation of the functions at seventeen points. The truncation error in the first order derivatives, Γ^{I}, (let Γ represent any one of the eight functions and let the I indicate the first differentiation with respect to either of the variables) was proportional to $\frac{h^4}{30}\Gamma^{V}$. The truncation error in the second order derivatives, Γ^{II}, was proportional to $\frac{h^4}{90}\Gamma^{VI}$ and the error in the third order derivative, Γ^{III}, was proportional to $\frac{h^2}{4}\Gamma^{V}$. The third order derivative suffered from the highest degree of truncation error. This truncation error could be reduced by decreasing the step size, h. However, it was found that roundoff error began to dominate when the step size was smaller than 0.01% of the variable magnitude. This established a practical limit on the accuracy of the differentiation process which reflected the balancing of the two opposing trends of truncation error versus roundoff error.

Method 2

Method 2 sought to eliminate the need for evaluating the nine partial derivatives of the eight functions by the method of taking differences. To do this it was necessary to use the definitions of the functions in Table 6-1 to obtain analytic expressions for all the derivatives in terms of the integrals over beta. This eliminates the serious limitations Method 1 had encountered on the precision of the third order derivatives, but it does so at the cost of introducing a host of new beta integrals to be evaluated. On the other hand the beta integrals do not have to be evaluated to the same extreme degree of precision that they had to be in Method 1 because the integrals were not later to be used to evaluate first, second and third order differences. The analytic expressions for all the derivatives are not provided here because they are lengthy. The expressions for the derivatives can be found, as they appear in the Mass Shell Program, in Appendix C.

Method 2 did not include special treatment for the many new beta integrals for the cases where there exists a β_0 such that $\frac{x_{\beta_0}}{u_{\beta_0}} = 1$ or $\frac{x_{\beta_0}}{u_{\beta_0}} = 0$. Therefore a test is made for each point to be calculated to determine if β_0 lies in the range from -1 to 1. In the event such a β_0 isfound, the rejection of the point is automatic when Gaussian quadrature is used. In those cases it is easier to return to Method 1 or switch to the slower Romberg integration than to try to reexpress all the new beta integrals so that β could cross the singularity.

Method 3

There is a third approach possible to the problem of accurately evaluating the functions in Table 6-1. When it is very important to know the functions with great accuracy, it is better to perform the integrations over beta analytically. The analytic expressions for the beta integrals are different for $\frac{k^2}{m^2} > 4$ and $\frac{k^2}{m^2} < 4$. Therefore it is necessary to work out the analytic expressions twice. This approach was first taken by Ruben Mendez Placito.[26] He worked out the analytic expressions for the eight scalar functions for $\frac{k^2}{m^2} < 4$. Then he applied a numerical procedure to evaluate the derivatives. By this method the value of the eight functions is precisely given, but the derivatives are subject to the same squaring off of truncation versus roundoff error which was mentioned in the discussion of Method 1. The leading terms on the L.H.S. of the vertex equations are the terms with the highest order derivatives. Therefore the error generated in creating the third order derivatives goes directly into the relative error of each equation. It is clear that the ultimate improvement upon all three methods would be to eliminate all the numerical processes of integration and differentiation. The improvement that Method 3 represents for the functions should be made also for all the derivatives. It is also clear that such a maneuver, done in duplicate for $\frac{k^2}{m^2} > 4$ and $\frac{k^2}{m^2} < 4$, would represent a considerable investment in effort.

7-4 Summary of Results of Mass Shell Program

The three methods just described were used to test the Mass Shell Solution. The expectation was that the Mass Shell Solution would provide a reasonable solution to the vertex equation in the vicinity of the mass shell. Using units which set the experimental mass of the electron to unity, the mass shell is given by the sphere described by $p^2 = 1$, $k^2 = 0$ and $-\infty < u < \infty$. This region was sampled. Because each of the three methods is associatd with a different kind of approximation, the differences in the results they yielded provided information about the importance of these approximations.

Around the mass shell, Method 1 provided a 1% error in the seven of the eight differential equations considered. The 1% error in the differential equations represents the accumulation of all the errors throughout both programs and therefore the systematic error in the functions can be taken to be less than 1%. In addition it was found that Method 1 could not approach the mass shell too closely. The error grows dramatically if p^2 is less than 1.01 and k^2 is less than 10^{-3}. At the point p^2 equals 1.0001, k^2 equals 1×10^{-12}, for all values of u, there is no resemblance of a solution remaining. It was not immediately clear whether this failure was due to the Mass Shell Solution itself or due to a loss of precision where it was most needed.

The explanation for the problem was evident when Method 2 was applied. At the same point that Method 1 totally broke down, p^2 equals 1.0001 and k^2 equals 1×10^{-12}, Method 2 yielded less than 2% error for the first four equations and less than 5% error for the last three

equations. The error climbs only slowly as p^2 approaches $1.0 + 1 \times 10^{-10}$ and k^2 approaches 1×10^{-16}. As the mass shell is approached the third order derivatives dominate the L.H.S. of the differential equation. Method 2 provides a better resolution of the third order derivatives and demonstrates that this is exactly what is needed to go arbitrarily close to $p^2 = 1$ and $k^2 = 0$. By this same reasoning Method 3 was not expected and was not found to perform well in the very immediate vicinity of the mass shell. However at the moderate range out from the mass shell it performed reliably.

It was interesting to discover that near the mass shell using Method 2 in conjunction with the Romberg subroutine, with a very high level of stability enforced, yielded the same results as the Gaussian subroutine. Apparently the integrals over beta were not contributing significant error in this region. The really important factor is control over the level of precision in the higher order derivatives.

The most surprising result that came out of testing the Mass Shell Solution was the way the solution still continued to work at points of intermediate and even large distances from the mass shell. Holding $k^2 = 10^{-3}$ and increasing p^2 from 1.0 to 1.0×10^{12} still found all three methods performing well. Method 3 generally had errors less than 1% while the other two methods hovered around 1%. Method 2 and Method 3 did not perform well for those values of u that caused the beta to be singular. However, Method 1 did perform properly for those values of u.

It was not possible to test the Mass Shell Solution for values of p^2 greater than 1.0×10^{12} because the size of numbers calculated exceeded the capacity of the machine and overflows resulted.

When p^2 was held constant and k^2 was allowed to grow, it was found that all three methods concurred that there is a natural upper bound on the allowable range of k^2. The Mass Shell Solution consistently fails when k^2 is greater than ten times p^2, for any value of p^2.

In conclusion, these trials had shown that the Mass Shell Solution is a solution to the vertex equation near the mass shell as it chould be. Perhaps even more importantly it was shown to have an unexpectedly large range of applicability covering the area,

$$1 < p^2 < 10^{12}$$

$$0 < k^2 < 10\ p^2$$

$$-\infty < u < \infty.$$

As a result of the testing we have a prescription for the method to be used to evaluate the Mass Shell Solution,

For the Mass Shell Region - Method 2, Gaussian Quadrature

For the Mass Shell Region with β integral singularities - Method 2, Romberg Integration

For the Region beyond the Mass Shell. - Method 1, Gaussian Quadrature

We would also predict that if Method 3 were extended so that, not only the functions were expressed analytically, but also all the derivatives, then it would out perform both of the earlier Methods in all regions.

The Mass Shell Solution has presented some further questions. How would the Mass Shell solution perform for values of p^2 greater than 1×10^{12}? Is there a way to test it beyond this upper bound?

Why does the solution begin to fail when k^2 is greater than 10 p^2?

The next chapter addresses itself to the exploration of these difficulties.

CHAPTER VIII

EXTENDING THE MASS SHELL SOLUTION

8-1 A Scaling Symmetry

The work described in Chapter VII found the Mass Shell Solution (M.S.S.) to be a viable solution to the vertex equations in the mass shell region and also extending over an unexpectedly large range in the magnitude of p^2. There remained two principal areas of concern. The first was, would the M.S.S. continue to be a solution where the magnitude of p^2 was greater than 10^{12}, or was the gradual incline in percent error (up to 3%), as p^2 approaches the machine imposed upper bound, an indication of the failure of the analytic approximation. The second concern was what was the cause of the upper bound on the magnitude of k^2 and would an understanding of the cause lead to a prescription for its resolution?

In the course of addressing these matters, the M.S.S. was modified to make the examination of the vertex equations more flexible. The vertex function is a dimensionless function of the electron momentum $\bar{p}$, the photon momentum $\bar{k}$ and the experimental mass m. For convenience, the mass is normally taken to be unity. It is possible to let the four-vector $\bar{k}$ define an axis of the coordinate system. The significant quantities then are the magnitude of $\bar{k}$ relative to the mass, the magnitude of $\bar{p}$ relative to $\bar{k}$ and the angle between $\bar{p}$ and $\bar{k}$, denoted u. The indefinite metric introduces the inconvenient feature that four-momenta squared need not

be positive definite. In all of the previously described work the problem was circumvented by considering only those values of p^2 and k^2 which were greater than zero. No difficulty was then encountered in evaluating u nor in evaluating the functions described in Table 6-1. However, a real advantage of allowing p^2 and k^2 to take on negative values simultaneously is that no singularities are encountered in the evaluation of the beta integrals. With no singularities present, the Gaussian method of integration could be used with impunity. Any failure of the M.S.S. to fit the vertex equations could not be attributed to an awkward estimation of the beta integrals and a more definitive test of the M.S.S. could be made.

The strategy employed to extend the M.S.S. into the range where the square of both momenta were negative was to regroup the products of all the momenta so that, nowhere, do the terms $p = (p^2)^{\frac{1}{2}}$ and $k = (k^2)^{\frac{1}{2}}$ appear. Two new terms were introduced called pk and pok to replace p times k and p divided by k.

$$p \times k \rightarrow pk \quad \text{where} \quad pk = \pm\, (p^2 k^2)^{\frac{1}{2}} \tag{8-1}$$

$$p/k \rightarrow pok \quad \text{where} \quad pok = (p^2/k^2)^{\frac{1}{2}} \tag{8-2}$$

The $\pm$ sign in the definition of the variable pk is taken to be positive when the electron and photon momenta squared are both positive. Conversely, the $\pm$ sign is taken to be negative when both squared momenta are negative. This reflects the fact that the product or quotient of pk and pok should recover the sign of the negative momenta when both p^2 and k^2 are negative.

Odd combinations of momenta occur in the F, H_0, H_1 and H_2 functions as they appear in Table 6-1. This is easily repaired by multiplying the

functions by an additional $(k^2)^{\frac{1}{2}}$. The addition of $(k^2)^{\frac{1}{2}}$ is compensated for by dividing the F, H_0, H_1 and H_2 functions by the same factor where they appear in the vertex equations. Thus the notation used to express the transverse component of the vertex function is changed in a small way to now be[†]

$$\begin{aligned}\Gamma^\lambda(\bar{p},\bar{p}+\bar{k}) = {} & \frac{p^\lambda}{pk}\,\bar{F}(p^2,u) + \gamma^\lambda \bar{G}_0(p^2,u) + \frac{p^\lambda \not{p}}{p^2}\,\bar{G}_1(p^2,u) \\ & + \frac{p^\lambda k}{pk}\,\bar{G}_2(p^2,u) + p^\lambda[\not{p},k]\,\frac{\bar{H}_0}{p^2k^2}(p^2,u) \\ & + [\gamma^\lambda,\not{p}]\,\frac{\bar{H}_1}{pk}(p^2,u) + [\gamma^\lambda,k]\,\frac{\bar{H}_2}{k^2}(p^2,u) \\ & + \varepsilon^{\lambda\alpha\nu\phi}\gamma^5\gamma_\phi p_\nu k_\alpha\,\frac{\bar{I}}{pk}(p^2,u). \end{aligned} \tag{8-3}$$

The expressions of the functions has also taken on a slightly different form. Table 6-1 is now to be replaced by Table 8-1 which includes this and other extensions, yet to be described.

No attempt has been made to generalize the Main and Mass Shell Programs to handle the alternate possibility of momenta of opposite sign, that is, $p^2 = -k^2$. It is not expected that this should incur any special obstacles other than the rather considerable initial inconvenience of tracking through the programs a large number of sign changes.

[†]The bar over the eight vertex functions is to remind the reader that the expressions in Table 8-1 should be used, not the earlier version of the expressions found in Table 6-1.

The next extension which was made was one that allows the study of points taken for $p^2/m^2 > 10^{12}$. Because the vertex function is dimensionless, it is invariant under scale transformations which treat equally the electron momentum, the photon momentum and the mass. Thus if a solution could be obtained for the coordinates

$$(p^2,\ k^2,\ m^2,\ u) = (101,\ 1 \times 10^{-3},\ 100,\ u), \tag{8-4}$$

this would be same as the solution at the point,

$$(p^2,\ k^2,\ m^2,\ u) = (1.01,\ 1 \times 10^{-5},\ 1.0,\ u). \tag{8-5}$$

It is the relative magnitudes of p^2, k^2 and m^2 which identify a point for some particular value of u. The angle variable, u, is independent of the scale changes. Although it is not possible to evaluate the vertex at $p^2 > 10^{12}$ because of the high powers of p^2 accumulated in the course of the Main Program, it is possible to let m^2 become arbitrarily small relative to p^2 and k^2. The mass appears only up to squared powers and doesn't threaten the capacity of the machine. The solution at the point

$$(p^2,\ k^2,\ m^2,\ u) = (1.0,\ 1 \times 10^{-10},\ 1 \times 10^{-80},\ u) \tag{8-6}$$

is equivalent to the point

$$(p^2, k^2, m^2, u) = (1 \times 10^{80},\ 1 \times 10^{70},\ 1.0,\ u). \tag{8-7}$$

By this strategy it is possible to test the M.S.S. at points which are equivalent to those which have p^2/m^2 attain the limiting value of 10^{80} or less.

This scaling symmetry was used to test the validity of the M.S.S. in the asymptotic p^2 region. It was found that the M.S.S. as represented by the functions in Table 6-1 loses its resemblance to a solution rapidly as the magnitude of p^2/m^2 grows larger than 10^{12}.

Before going on to describe the work that was done to correct the difficulty at asymptoticly large p^2, it may be worth mentioning in passing that the scaling symmetry provided a handy diagnostic tool for detecting bugs. After any major alteration of either the Main Program or the M.S.S. Program it was found useful to test them out at a set of equivalent points. Even small and hard to find errors were unequally treated at such equivalent points and their occurrence could be readily detected. This tool was routinely applied after any major changes in the programs.

8-2 The Large p^2 Region

The deterioration of the accuracy of the M.S.S. for large p^2 begs for a re-examination of the premises upon which its derivation was based. In arriving at the solutions described in Table 6-1 it was assumed that γ^λ was the dominant term in Γ^λ. It was assumed that all other terms were smaller by at least one order of magnitude of the fine structure constant. For large values of p^2 (relative to k^2 and u^2), this approximation is not in contradiction with the value of the vertex function obtained from the M.S.S. In fact, as p^2 grows very large, all the functions except G_0 decrease at a rate of $1/p$ or faster. The function, G_0, approaches a constant equal to unity. Therefore the approximation used to derive the M.S.S. not only holds but is actually better in the asymptotic p^2 region

than it is in the mass shell region. By this criterion alone the M.S.S. should be expected to work in the asymptotic p^2 region.

However, in deriving the M.S.S. another approximation was made. It was assumed

$$A(p_1^2) = A(p_2^2) = m, \tag{8-8}$$

where m is the experimental mass. This approximation is completely wrong in the large p^2 region since as

$$|p^2| \to \infty, \quad A(p_1^2) \to 0 \quad \text{and } A(p_2^2) \to 0. \tag{8-9}$$

In deriving the M.S.S. as it appears in Table 6-1, the approximation in Eq. (8-8) was used to provide a simplified expression of the D^λ, D^λ_μ, $D^\lambda_{\mu\nu}$ and $D^\lambda_{\mu\nu\rho}$ tensors (first defined in Eq. (6-7)). If the approximation had not been employed the tensors, D^λ, D^λ_μ, $D^\lambda_{\mu\nu}$ and $D^\lambda_{\mu\nu\rho}$, would have taken the form,

$$
\begin{aligned}
D^\lambda &= [A_1 p_2^\lambda + A_2 p_1^\lambda]/D_{12} \\
D^\lambda_\mu &= [(A_1 A_2 - \bar{p}_1\ \bar{p}_2)\delta^\lambda_\mu + (p_{1\mu} + p_{2\mu})p^\lambda]/D_{12} \\
D^\lambda_{\mu\nu} &= [A_2 p_{1\nu} - A_1 p_{2\nu})\delta^\lambda_\mu - (A_2 p_{1\mu} - A_1 p_{2\mu})\delta^\lambda_\nu]/D_{12} \\
D^\lambda_{\mu\nu\rho} &= [(p_{1\mu} p_{2\nu} - p_{2\mu} p_{1\nu})\delta^\lambda_\rho + (p_{1\nu} p_{2\rho} - p_{2\nu} p_{1\rho})\delta^\lambda_\mu \\
&\quad + (p_{1\rho} p_{2\mu} - p_{2\rho} p_{1\mu})\delta^\lambda_\nu]/D_{12}
\end{aligned} \tag{B-11}
$$

where $D_{12} = (p_1^2 - A_1^2)(p_2^2 - A_2^2)$ and $A_1 = A(p_1^2)$, $A_2 = A(p_2^2)$ instead of that given by Eq. (6-11).

These new expressions for the D^λ, D^λ_μ, $D^\lambda_{\mu\nu}$ and $D^\lambda_{\mu\nu\rho}$ tensors lead to a new and necessarily more complicated set of differential equations which have to be solved to define the C^λ, C^λ_μ, $C^\lambda_{\mu\nu}$ and $C^\lambda_{\mu\nu\rho}$ tensors, which in turn, would lead to an identification of the $\overline{F}$, $\overline{G}_0$, $\overline{G}_1$, $\overline{G}_2$, $\overline{H}_0$, $\overline{H}_1$, $\overline{H}_2$ and $\overline{I}$ functions. As an example of how this less restricted form of the D^λ tensor affects the problem, consider the new equation for the C^λ tensor which leads to the identification of the $\overline{F}$ function,

$$\nabla^2 C^\lambda = - 3\varepsilon\left[\frac{A_1 p_1^\lambda}{D_{12}} - \frac{A_2 p_2^\lambda}{D_{12}}\right] . \tag{8-10}$$

It is impossible to solve this equation by the same methods employed in Chapter VI and Appendix B. The transformation functions given in Eqs. (B-21) through (B-30) no longer apply.

An alternative to resolving these more complicated expressions for the eight vertex functions is suggested by the observation that the mass function, $A(p^2)$, is slowly changing. The derivatives of $A(p_1^2)$ and $A(p_2^2)$ are very small and $A(p_1^2)$ and $A(p_2^2)$ will behave much like constants over a finite range of p^2. Furthermore $A(p_1^2)$ is approximately equal to $A(p_2^2)$ where the ratio, p^2/k^2, is large. A useful adulteration of the M.S.S. would be to replace m^2 by A_1A_2 and m by $(A_1+A_2)/2$, wherever they appear in the definition of the functions. This crude recasting would not be expected to be favorable in any region where $k^2 \gg p^2$ (because $A(p_1^2) \neq A(p_2^2)$). However, it has already been found that the M.S.S. is not valid where $k^2 > p^2$, so no loss will be suffered on that account. Furthermore, since as the mass shell is approached, the mass function

approaches the value of the mass, the original M.S.S. will be recovered and no harm will be done to the solution in the region of the mass shell.

Some tinkering was done with this fomula. The best results were obtained when the replacement of m^2 by A_1A_2 and m by $(A_1+A_2)/2$ was made almost everywhere. The exception lay in the CK function which appears in Table 8-1 and which defines the vertex function G_0. The m^2 in the natural logarithm is not altered because when it was altered it introduced a small error which grew in inverse proportion to the scale of the mass. The changes that were made are summarized in Table 8-1. Table 8-1 represents the Extended Mass Shell Solution (E.M.S.S.).

The E.M.S.S. performs dramatically better than the M.S.S. Figure 8-1 has been prepared to illustrate how extensive the solution is. For any relative value of k^2 to m^2 the graph plots the dependence of the solution on the variables p^2/k^2 and u^2k^2/p^2. The error is less than three percent in the region bounded on the right by $u^2k^2/p^2 = 1 \times 10^5$ and on the bottom by $p^2/k^2 = 1 \times 10^{-3}$. However, when the approach to the mass shell is extremely close there is a rise in the error in the sixth and seventh vertex equations which was attributed in Chapter 7 to the difficulty in obtaining extremely accurate third order derivatives in the function G_0. Otherwise the functions represent a very regular and predictable solution. The solution is marred by only the limitation that $u^2k^2/p^2 < 10^5$ and $p^2/k^2 > 10^{-3}$.

TABLE 8-1 THE EXTENDED MASS SHELL FUNCTIONS

$$\overline{F} = \frac{3}{8}\,\varepsilon(A_1+A_2)\ell\int I_1\, d\beta$$

$$\overline{I} = \frac{\varepsilon}{4}\,\ell\int I_1\, d\beta$$

$$\overline{G}_o = 1 + \frac{\varepsilon}{4}\int I_2\, d\beta - \frac{\varepsilon}{4}\, CK - \frac{\varepsilon}{4}\int I_3\, d\beta$$

$$\overline{G}_1 = \frac{\varepsilon}{2}\, p^2\int I_3^x\, d\beta$$

$$\overline{G}_2 = \frac{\varepsilon}{4}\,(2p^2u+\ell)\left[\int I_3^x d\beta + \int I_4^x d\beta\right] - \frac{\varepsilon}{4}\,\ell\int I_3^x d\beta$$

$$\overline{H}_o = -\frac{\varepsilon}{16}\,(A_1+A_2)k^2p^2\int I_5^{xx}\, d\beta$$

$$\overline{H}_1 = \frac{\varepsilon}{32}\,(A_1+A_2)\left[-\ell k^2\int I_5^{xx}\beta d\beta - 2\left(p^2k^2u + \frac{k^2\ell}{2}\right)\int I_5^{xx}\, d\beta\right]$$

$$\overline{H}_2 = \frac{\varepsilon}{16}(A_1+A_2)k^2\left[\int I_5^x d\beta + \frac{1}{2}\int I_6 d\beta\right]$$

$$+ \frac{\varepsilon}{16}\,(A_1+A_2)k^2\left(\frac{\ell u}{2} + \frac{k^2}{4}\right)\left[\int I_5^{xx}d\beta + \int I_5^{xx}\beta d\beta\right]$$

$$+ \frac{\varepsilon}{64}\, k^4(A_1+A_2)\int I_5^{xx}\beta^2 d\beta$$

$$\ell = (p^2k^2)^{\frac{1}{2}}$$

$$u = \frac{p\cdot k}{(p^2\cdot k^2)^{\frac{1}{2}}}$$

$$I_1 = \frac{1}{x_\beta}\left[\frac{\mu_\beta}{x_\beta}\ln\left(1 - \frac{x_\beta}{\mu_\beta}\right) + 1\right]$$

$$I_2 = \left(1 - \frac{\mu_\beta}{x_\beta} - \frac{k^2}{2x_\beta}\right)\ln\left(1 - \frac{x_\beta}{\mu_\beta}\right)$$

$$I_3 = (A_1A_2-\mu_\beta)\,\frac{1}{x_\beta}\left[\left(\frac{\mu_\beta}{x_\beta} - 1\right)\ln\left(1 - \frac{x_\beta}{\mu_\beta}\right) + 1\right]$$

TABLE 8 -1 (Continued)

$$I_4=(A_1A_2-\mu_\beta)\,\frac{1}{x_\beta}\ln\left(1-\frac{x_\beta}{u_\beta}\right)$$

$$CK=\left(1-\frac{A_1A_2}{p_1^2}\right)\ln\left(1-\frac{p_1^2}{m^2}\right)+\left(1-\frac{m^2}{p_2^2}\right)\ln\left(1-\frac{p_2^2}{m^2}\right)$$

$$I_3^x=(A_1A_2-\mu_\beta)\left[-\frac{2\mu_\beta}{x_\beta^2}\ln\left(1-\frac{x_\beta}{\mu_\beta}\right)+\frac{1}{x_\beta^2}\ln\left(1-\frac{x_\beta}{\mu_\beta}\right)-\frac{2}{x_\beta^2}\right]$$

$$I_4^x=(A_1A_2-\mu_\beta)\left[-\frac{1}{x_\beta^2}\ln\left(1-\frac{x_\beta}{\mu_\beta}\right)-\frac{1}{\mu_\beta x_\beta}\,\frac{1}{\left(1-\frac{x_\beta}{\mu_\beta}\right)}\right]$$

$$I_5=L_2\left(\frac{x_\beta}{\mu_\beta}\right)+\left(1-\frac{\mu_\beta}{x_\beta}\right)\ln\left(1-\frac{x_\beta}{u_\beta}\right)$$

$$L_2(z)=-\int_0^z\ln(1-z)\,\frac{dz}{z}$$

$$I_6=\frac{1}{x_\beta}\ln\left(1-\frac{x_\beta}{\mu_\beta}\right)$$

$$I_5^x=\frac{1}{\mu_\beta}\left[-\frac{\mu_\beta}{x_\beta}\ln\left(1-\frac{x_\beta}{\mu_\beta}\right)+\frac{u_\beta^2}{x_\beta^2}\ln\left(1-\frac{x_\beta}{\mu_\beta}\right)+\frac{\mu_\beta}{x_\beta}\right]$$

$$I_5^{xx}=\frac{1}{\mu_\beta^2}\left[\frac{\mu_\beta^2}{x_\beta^2}\ln\left(1-\frac{x_\beta}{\mu_\beta}\right)+\frac{\mu_\beta}{x_\beta}\,\frac{1}{\left(1-\frac{x_\beta}{\mu_\beta}\right)}-2\,\frac{\mu_\beta^3}{x_\beta^3}\ln\left(1-\frac{x_\beta}{\mu_\beta}\right)\right.$$
$$\left.-\frac{\mu_\beta^2}{x_\beta^2}\,\frac{}{\left(1-\frac{x_\beta}{\mu_\beta}\right)}-\frac{\mu_\beta^2}{x_\beta^2}\right]$$

where $x_\beta=[\bar{p}+\frac{1}{2}(\beta+1)\bar{k}]^2$

$\mu_\beta=A_1A_2-\frac{1}{4}(1-\beta^2)k^2$

$\varepsilon=\frac{\varepsilon}{\pi}$

α = fine structure constant

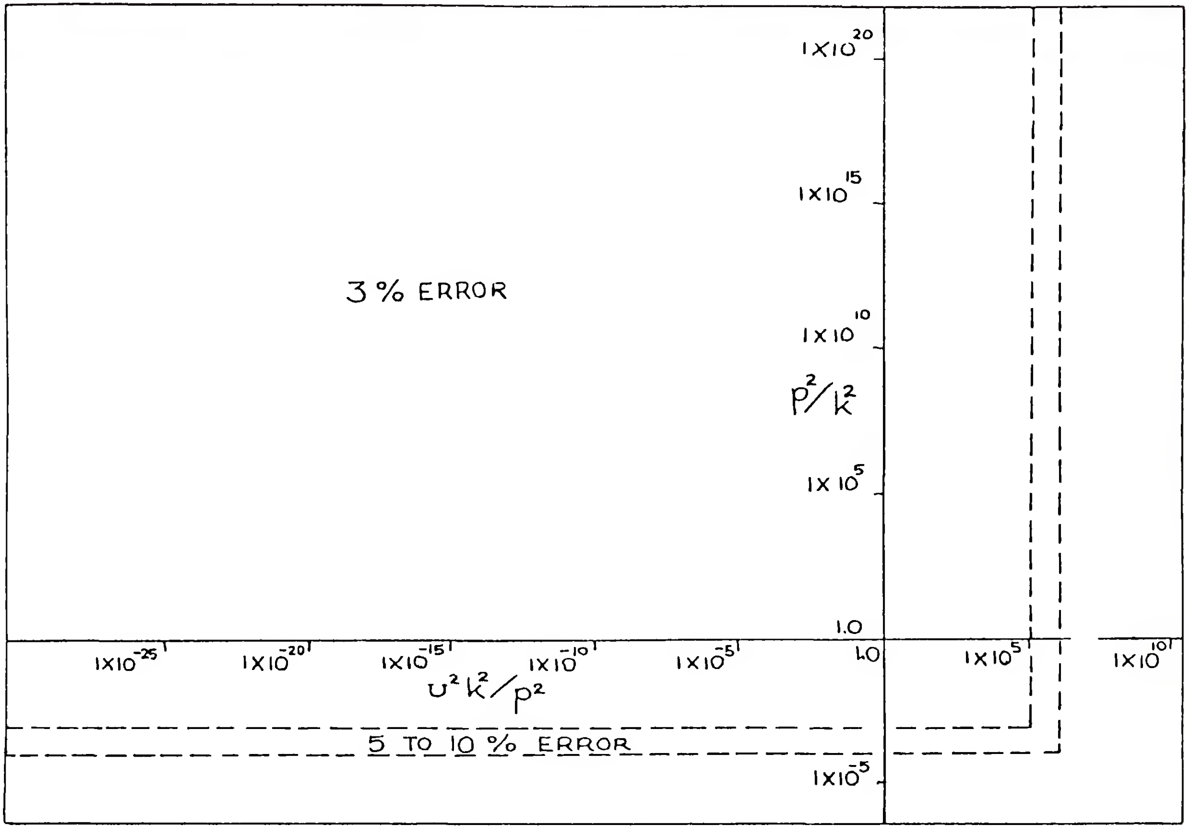

Figure 8-1 The Extended Mass Shell Solution

8-3 The Large k^2 Region

It is not difficult to see that the solution represented by either Table 6-1 or Table 8-1 is not going to be a valid solution to the vertex equations when the magnitude of k^2 grows very large relative to the magnitude of p^2. A necessary condition for the self consistency of the M.S.S. is that it should be dominated by the leading term $G_0 = 1$. However, since the functions F, I, H_0, H_1, H_2 and G_2 are proportional to the first and second powers of k, it is indicated that for fixed p^2 and sufficiently large k, these functions will grow to dominate over the contribution from $G_0 = 1$. Actual determinations of the M.S.S. in the large k^2 region (that region where $k^2 > 10^3 p^2$) do show these functions are much larger than $G_0 = 1$. In addition, when the M.S.S. was extended by the replacement of m by $(A_1+A_2)/2$ it was assumed that p_1^2 was not greatly different from p_2^2. This corresponds to assuming $|u^2k^2| < |p^2|$ and $|k^2| < |p^2|$. For these reasons, it is easy to see that neither the M.S.S. nor the E.M.S.S. should perform properly in any region where $|u^2k^2| >> |p^2|$ or $|k^2| >> |p^2|$.

The correction to the problem is easily proposed though the actual achievement of the correction may be very much more difficult. To correct the problem would necessitate returning to the beginning and assuming some approximate form for the large k^2 behavior of the functions F, I, H_0, H_1, H_2 and G_2. The differential equations from this approximation will be too complicated to solve directly. As was done in Chapter VI and Appendix B, the closest approximation to the real equations which is solvable would have to be used as an initial guess. Finally the new functions would

have to satisfy the boundary condition that they should smoothly join the functions in the E.M.S.S.

Basicly the problem is that the solution that we have, has built into it certain expectations about the relative magnitudes of the variables. The solution could be said to have a well-defined region of convergence. A trivial example is useful to illustrate this difficulty. Consider a function, f, of two variables p and q, given by

$$f(p,q) = \frac{1}{2p-3q}. \qquad (8\text{-}11)$$

Occasionally a function of the sort is needed in the form of a series expansion,

$$f(p,q) = \frac{1}{2p}\left[1 + \frac{3q}{2p} + \left(\frac{3q}{2p}\right)^2 + \ldots\right]. \qquad (8\text{-}12)$$

Some finite number of terms in the expansion will represent the function to a particular level of accuracy as long as $\frac{q}{p} < \frac{2}{3}$. Thus, f(2,1) could be calculated by Eq. (8-14) but f(1,2) could not be. If the value of f(1,2) was desired, it is necessary to return to the basic function, Eq. (8-13) and rewrite the expansion so that,

$$f(p,q) = -\frac{1}{3q}\left[1 + \left(\frac{2p}{3q}\right) + \left(\frac{2p}{3q}\right)^2 + \ldots\right]. \qquad (8\text{-}13)$$

In this illustration it is of course exceedingly simple to write out the second expansion. If the second expansion was harder to come by, it might be faster to look for a symmetry in the function in Eq. (8-13) which could provide a relationship which could predict the value of f(1,2) in terms of the first expansion. It is clear that the right-hand side of

Eq. (8-13) is unaffected under a transformation which replaces p by $\frac{3}{2}$ q and q by $\frac{2}{3}$ p and reverses the sign of the whole expression. Thus,

$$f(p,q) = - f(\tfrac{3}{2} q, \tfrac{2}{3} p), \tag{8-14}$$

and therefore, $f(1,2) = - f(3,2/3)$. The value of $f(1,2)$ may be obtained by evaluating the negative of $f(3,2/3)$. The value of $f(3,2/3)$ can be obtained from the first expansion. It can be seen then that if the second expansion is for some reason difficult to procure, the first expansion plus a symmetry relation will serve the same purpose.

It would be most useful if some kind of symmetry could be found to evaluate the vertex functions, outside of the region of three per cent error, in terms of the functions at a point inside the region of three percent error. With this thought in mind consider the relative magnitudes of the incoming and outgoing electron momenta of the vertex function $\Gamma^{\mu}(\bar{p},\bar{q})$, where $\bar{q} = \bar{p} + \bar{k}$. The incoming electron momentum squared is p^2, and as long as $p^2/k^2 > 10^{-3}$ and $u^2k^2/p^2 < 10^5$, $\Gamma^{\mu}(\bar{p},\bar{q})$ can be determined from Table 8-1. Yet the outgoing momentum, squared, $q^2 = p^2 + 2\bar{p}\cdot\bar{k} + k^2$, is not restricted like the incoming momentum,

$$0 \leqslant \frac{q^2}{k^2} < \infty. \tag{8-15}$$

If there were a symmetry relation between $\Gamma^{\mu}(\bar{q},\bar{p})$ and $\Gamma^{\mu}(\bar{p},\bar{q})$, then the vertex could be evaluated for cases where the incoming momentum squared was less than $10^{-3}k^2$.

There is an intuitive feeling that there should be a relationship between the amplitude for a process that goes in with momentum $\bar{p}$ and

out with momentum $\bar{q}$ and the amplitude for a process that goes in with momentum $\bar{q}$ and out with momentum $\bar{p}$.

In a report by A. A. Broyles,[27] the symmetries of the vertex function were exhaustibly analyzed. He used the following abbreviated notation for the complete vertex,[†]

$$\Gamma^{\mu}_{LT}(\bar{p},\bar{q}) = F^{\mu}(\bar{p},\bar{q}) + \gamma^{\nu}G^{\mu}_{\cdot\nu}(\bar{p},\bar{q}) + \sigma^{\nu\eta}H^{\mu}_{\cdot\nu\eta}(\bar{p},\bar{q})$$
$$+ i\gamma^{5}\gamma^{\nu}I^{\mu}_{\cdot\nu}(\bar{p},\bar{q}). \tag{8-16}$$

For comparison, the complete vertex from Eq. (2-17) is repeated here with the notation consistent with the use of Table 8-1 to define the eight transverse scalar functions, and consistent with Ward's Identity having been applied to identify the longitudinal scalar functions.

$$\Gamma^{\mu}_{LT}(\bar{p},\bar{q}) = \frac{p^{\mu}}{pk}\,\bar{F}(p^2,k^2,u,m) + \gamma^{\mu}\bar{G}_0(p^2,k^2,u,m)$$
$$+ \frac{p^{\mu}\not{p}}{p^2}\,\bar{G}_1(p^2,k^2,u,m) + \frac{p^{\mu}\not{k}}{pk}\,\bar{G}_2(p^2,k^2,u,m)$$
$$- 2ip^{\mu}\sigma^{\alpha\beta}p_{\alpha}k_{\beta}\,\frac{\bar{H}_0}{p^2k^2}(p^2,k^2,u,m) - 2i\sigma^{\mu\alpha}p_{\alpha}\,\frac{\bar{H}_1}{pk}(p^2,k^2,u,m)$$
$$- 2i\sigma^{\mu\alpha}k_{\alpha}\,\frac{\bar{H}_2}{k^2}(p^2,k^2,u,m) + \varepsilon^{\mu\alpha\nu\phi}\gamma^{5}\gamma_{\phi}p_{\nu}k_{\alpha}\,\frac{\bar{I}}{pk}(p^2,k^2,u,m)$$
$$+ \frac{k^{\mu}}{k^2}\,[A(q^2) - A(p^2)] - \frac{k^{\mu}\not{q}}{k^2} + \frac{k^{\mu}\not{p}}{k^2}\,. \tag{8-17}$$

[†]Because the reader has become accustomed to notation representing the transverse vertex by Γ^{μ}, the designation Γ_{LT} will be used to signify the complete vertex equation.

For future reference we will equate Eqs. (8-16) and (8-17) and make the following identifications.

$$F^{\mu}(\bar{p},\bar{q}) = \frac{p^{\mu}}{pk}\,\bar{F}(p^2,k^2,u,m) + \frac{k^{\mu}}{k^2}\,[A(q^2) - A(p^2)]$$

$$G^{\mu}_{\cdot\nu}(\bar{p},\bar{q}) = g^{\mu}_{\cdot\nu}\bar{G}_o(p^2,k^2,u,m) + \frac{p^{\mu}p_{\nu}}{p^2}\,\bar{G}_1(p^2,k^2,u,m)$$

$$+ \frac{p^{\mu}k_{\nu}}{pk}\;\bar{G}_2(p^2,k^2,u,m) - \frac{k^{\mu}q_{\nu}}{k^2} + \frac{k^{\mu}p_{\nu}}{k^2}$$

$$H^{\mu}_{\cdot\nu\eta}(\bar{p},\bar{q}) = -\,2ip^{\mu}p_{\nu}k_{\eta}\,\frac{\bar{H}_o}{p^2k^2}\,(p^2,k^2,u,m) - 2ig^{\mu}_{\nu}p_{\eta}\,\frac{\bar{H}_1}{pk}\,(p^2,k^2,u,m)$$

$$-\,2ig^{\mu}_{\cdot\nu}k_{\eta}\,\frac{\bar{H}_2}{k^2}\,(p^2,k^2,u,m)$$

$$I^{\mu}_{\cdot\nu}(\bar{p},\bar{q}) = -\,i\varepsilon^{\mu\alpha\beta}{}_{\nu}p_{\beta}k_{\alpha}\,\frac{\bar{I}}{pk}\,(p^2k^2u,m). \qquad (8\text{-}18)$$

Having established the new notation in relation to the more familiar one we now examine the symmetries of Eq. (8-16).

Under charge conjugation of the γ^{μ} matrix transforms in the following way,

$$(C\gamma^o)\gamma^{\mu *}(C\gamma^o)^{-1} = -\,\gamma^{\mu}, \qquad (8\text{-}19)$$

where $\quad C = i\gamma^2\gamma^o,$

and since $\gamma^o\gamma^{\mu *}\gamma^o = \gamma^{\mu T}$ (where T represents the transpose operation),

$$C^{-1}\gamma^{\mu}\,C = -\,\gamma^{\mu T}. \qquad (8\text{-}20)$$

The vertex function transforms under charge conjugation in the same fashion as γ^{μ} does,

$$C^{-1}\Gamma^{\lambda}_{LT}\,C = -\,\Gamma^{\lambda T}_{LT}\,. \qquad (8\text{-}21)$$

This places the following restrictions on the tensor functions F^{μ}, $G^{\mu}_{\cdot\nu}$, $H^{\mu}_{\cdot\nu}$ and $I^{\mu}_{\cdot\nu}$,

$$F^{\mu}(-\bar{p},-\bar{q})^{*} = - F^{\mu}(\bar{p},\bar{q}) \tag{8-22}$$

$$G^{\mu}_{\cdot\nu}(-\bar{p},-\bar{q})^{*} = G^{\mu}_{\cdot\nu}(\bar{p},\bar{q}) \tag{8-23}$$

$$H^{\mu}_{\cdot\nu\eta}(-\bar{p},-\bar{q})^{*} = H^{\mu}_{\cdot\nu\eta}(\bar{p},\bar{q}) \tag{8-24}$$

$$I^{\mu}_{\cdot\nu}(-\bar{p},-\bar{q})^{*} = - I^{\mu}_{\cdot\nu}(\bar{p},\bar{q}). \tag{8-25}$$

The γ^{μ} matrices are self-adjoint and the vertex function will transform like γ^{μ} so that

$$\bar{\Gamma}^{\mu}_{LT}(\bar{p},\bar{q}) = \gamma^{o}\Gamma^{\mu}_{LT}(\bar{q},\bar{p})^{*T}\,\gamma^{o} = \Gamma^{\mu}_{LT}(\bar{p},\bar{q}) \tag{8-26}$$

where the bar over Γ^{μ} stands for the adjoint operation. When the adjoint operation is applied to Eqs. (8-22) through (8-25) the following relations are obtained,

$$F^{\mu}(\bar{p},\bar{q}) = - F^{\mu}(-\bar{q},-\bar{p}) \tag{8-27}$$

$$G^{\mu}_{\cdot\nu}(\bar{p},\bar{q}) = G^{\mu}_{\cdot\nu}(-\bar{q},-\bar{p}) \tag{8-28}$$

$$H^{\mu}_{\nu\eta}(\bar{p},\bar{q}) = H^{\mu}_{\cdot\nu\eta}(-\bar{q},-\bar{p}) \tag{8-29}$$

$$I^{\mu}_{\cdot\nu}(\bar{p},\bar{q}) = -I^{\mu}_{\cdot\nu}(-\bar{q},-\bar{p}). \tag{8-30}$$

Thus

$$\Gamma^{\mu}_{LT}(\bar{q},\bar{p}) = - F^{\mu}(-\bar{p},-\bar{q}) + \gamma^{\nu}G^{\mu}_{\cdot\nu}(-\bar{p},-\bar{q})$$

$$+ \sigma^{\nu\eta}H^{\mu}_{\cdot\nu\eta}(-\bar{p},-\bar{q}) - i\gamma^{5}\gamma^{\nu}I^{\mu}_{\cdot\nu}(-\bar{p},-\bar{q}). \tag{8-31}$$

Eq. (8-18) can be used to identify the tensors on the right of Eq. (8-31) in terms of the scalar functions. We find

$$\Gamma^{\mu}_{LT}(\bar{q},\bar{p}) = \frac{p^{\mu}}{pk}\,\bar{F}(p^2,k^2,u,m) + \frac{k^{\mu}}{k^2}\,[A(q^2) - A(p^2)]$$

$$+ \gamma^{\mu}\bar{G}_o(p^2,k^2,u,m) + p^{\mu}\not{p}\,\frac{\bar{G}_1}{p^2}\,(p^2,k^2u,m)$$

$$+ \frac{p^{\mu}\not{k}}{pk}\,\bar{G}_2(p^2,k^2,u,m) - \frac{k^{\mu}\not{q}}{k^2} + \frac{k^{\mu}\not{p}}{k^2}$$

$$+ 2i\sigma^{\alpha\beta}p_{\alpha}k_{\beta}p^{\mu}\,\frac{\bar{H}_o}{p^2k^2}\,(p^2,k^2,u,m)$$

$$+ 2i\sigma^{\mu\alpha}p_{\alpha}\,\frac{\bar{H}_1}{pk}\,(p^2,k^2,u,m) + 2i\sigma^{\mu\alpha}k_{\alpha}\,\frac{\bar{H}_2}{k^2}\,(p^2,k^2,u,m)$$

$$- \varepsilon^{\mu\alpha\beta\nu}\,\gamma^5\gamma_{\nu}p_{\beta}k_{\alpha}\,\frac{\bar{I}}{pk}\,(p^2,k^2,u,m). \qquad (8\text{-}32)$$

We return to our principal interest in the transverse components of the vertex only. Furthermore, because Figure 8-1 holds for any ratio of $|k^2|$ to m^2 the eight transverse scalar functions are identified by the variables which form the axis of graph in Figure 8-1, namely, p^2/k^2 and u^2k^2/p^2 plus a statement of the magnitude of k^2/m^2. Therefore, we will write the transverse vertex as

$$\Gamma^{\mu}(\bar{q},\bar{p}) = \frac{p^{\mu}}{pk}\,\bar{F}(p^2/k^2,u^2k^2/p^2) + \gamma^{\mu}\bar{G}_o(p^2/k^2,u^2k^2/p^2)$$

$$+ p^{\mu}\not{p}\,\frac{\bar{G}_1}{p^2}\,(p^2/k^2,u^2k^2/p^2) + p^{\mu}k\,\frac{\bar{G}_2}{pk}\,(p^2/k^2,u^2k^2/p^2)$$

$$- p^{\mu}[\not{p},\not{k}]\,\frac{\bar{H}_o}{p^2k^2}\,(p^2/k^2,u^2k^2/p^2)$$

$$- [\gamma^\mu,\not{p}]\ \frac{\bar{H}_1}{pk}\ (p^2/k^2, u^2k^2/p^2)\ -\ [\gamma^\mu,k]\ \frac{\bar{H}_2}{k^2}\ (p^2.k^2,u,m)$$

$$- \varepsilon^{\mu\alpha\beta\nu}\ \gamma^5\gamma_\nu p_\beta k_\alpha\ \frac{\bar{I}}{pk}\ (p^2/k^2, u^2k^2/p^2) \qquad (8\text{-}33)$$

for $\bar{q} = \bar{p} + \bar{k}$ and for some fixed k^2/m^2.

A comparison with Eq. (8-3) reveals that the difference between the expressions $\Gamma^\mu(\bar{p},\bar{q})$ and $\Gamma^\mu(\bar{q},\bar{p})$ is only that the signs of the $\bar{H}_o$, $\bar{H}_1$, $\bar{H}_2$ and $\bar{I}$ functions have been reversed.

$\Gamma^\mu(\bar{q},\bar{p})$ is a function of the variables q^2/k^2 and $u_q^2k^2/q^2$ for a particular k^2/m^2.

$$u_q = \frac{\bar{q}\cdot\bar{k}}{(q^2)^{\frac{1}{2}}(k^2)^{\frac{1}{2}}}\ . \qquad (8\text{-}34)$$

It has been related to a particular combination of the eight scalar functions which are defined by the coordinates p^2/k^2, u^2k^2/p^2 for the same k^2/m^2. Since $\bar{q} = \bar{p} + \bar{k}$, it follows that

$$\frac{q^2}{k^2} = \frac{p^2}{k^2} + 2\left(\frac{u^2p^2}{k^2}\right)^{\frac{1}{2}} + 1 \qquad (8\text{-}35)$$

and

$$\frac{u_q^2k^2}{q^2} = [(u^2p^2/k^2)^{\frac{1}{2}} + 1]^2\left(\frac{k^2}{q^2}\right)^2. \qquad (8\text{-}36)$$

We would like to show that any value of q^2/k^2 and $u_q^2k^2/q^2$ outside of the three percent error region maps onto a point $(p^2/k^2, u^2k^2/p^2)$ in the interior of the three percent error region. To do this we will examine three cases. In the first case we can select $\bar{q}^2/k^2$ to have some

arbitrary value and show that $u_q^2k^2/q^2$ increasing from 10^5 to infinity maps to a line of $(p^2/k^2,\ u^2k^2/p^2)$ points in the one percent region. Consider Figure 8-2 which is a replica of Figure 8-1. The vertical axis describes the ratio of the incoming momentum squared to the photon momentum squared. The horizontal axis is the quotient of the square of the angle variable for the incoming momentum and vertical axis variable. The shaded area covers the three percent error region. For our example, let $q^2/k^2 = b^2$ and let $u_q^2k^2/q^2 = 10^5a^2$. When a^2 is allowed to vary from one to infinity a line of points is drawn on the graph, denoted by the arrow.

Next we solve for p^2/k^2 and u^2k^2/p^2,

$$p^2/k^2 = b^2 \mp 2(a^2 10^5)^{\frac{1}{2}}\ b^2 + 1$$

and

$$u^2k^2/p^2 = \frac{[(a^2\ 10^5)^{\frac{1}{2}}\ b^2 - 1]^2}{[b^2 + 2(a^2\ 10^5)^{\frac{1}{2}}\ b^2 - 1]^2}\ .$$

For any fixed value of b, varying a from -1 to $-\infty$ draws a line of points with increasing p^2/k^2 and u^2k^2/p^2 approaching 1/4. This line is the new arrow that appears in Figure 8-3.

By a similar analysis it can be shown that for a fixed value of $u_q^2k^2/p^2 = b^2$ and a value of $q^2/k^2 = 10^{-3}/a^2$ where a ranges from 1 to ∞ a line is drawn from the lower boundary of the one percent region. The line appears in Figure 8-4, pointing vertically down. The infinite line of $(q^2/k^2,\ u_q^2k^2/q^2)$ points is mapped onto a short finite line of $(p^2/k^2,\ u^2k^2/p^2)$ points which approach the value (1,1).

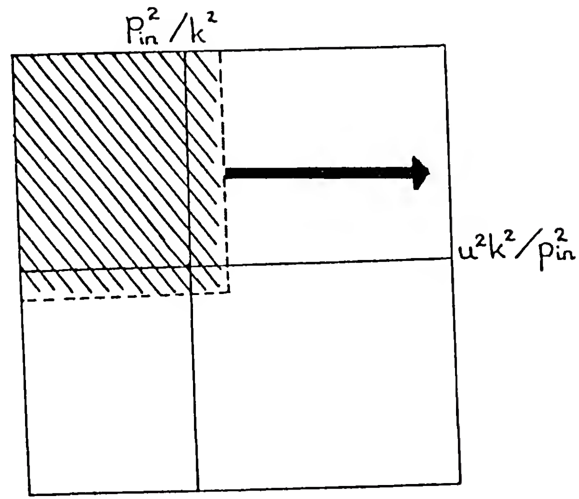

FIGURE 8-2 The Variation of a

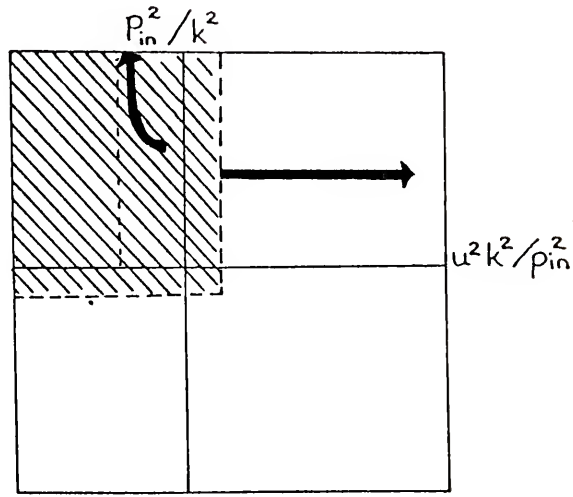

FIGURE 8-3 The Mapping of the Variation of a

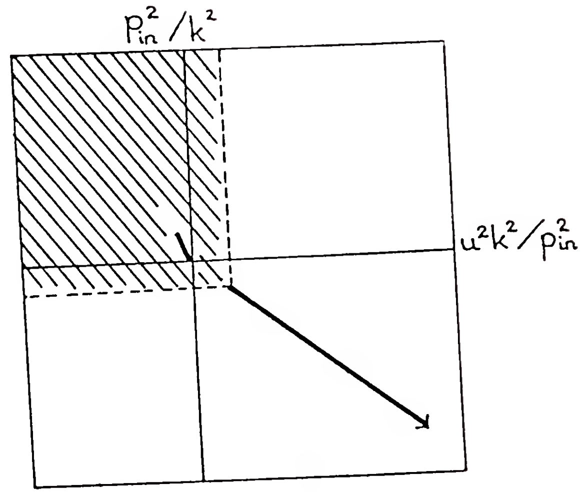

FIGURE 8-4 The Mapping of the Variation of b^2

By repeating this process of drawing horizontal lines for b^2 fixed and vertical oines for a fixed and negative, it is possible to see how the $(q^2/k^2,\ u_q^2k^2/q^2)$ region maps onto the $(p^2/k^2,\ u^2k^2/p^2)$ region. In Figure 8-5 an illustration is given of this relationship.

In conclusion we find that Table 8-1 gives a description of the solution to the transverse vertex equation for the region $p^2/k^2 > 10^{-3}$ and $u^2k^2/p^2 < 10^5$. It is possible to evaluate the solution outside of the region by the relation given in Eq. (8-33) for $\Gamma^\mu(q,p)$ where the coordinates of incoming momentum q are related to those of an incoming momentum $\bar{p}$ by Eqs. (8-35) and (8-36).

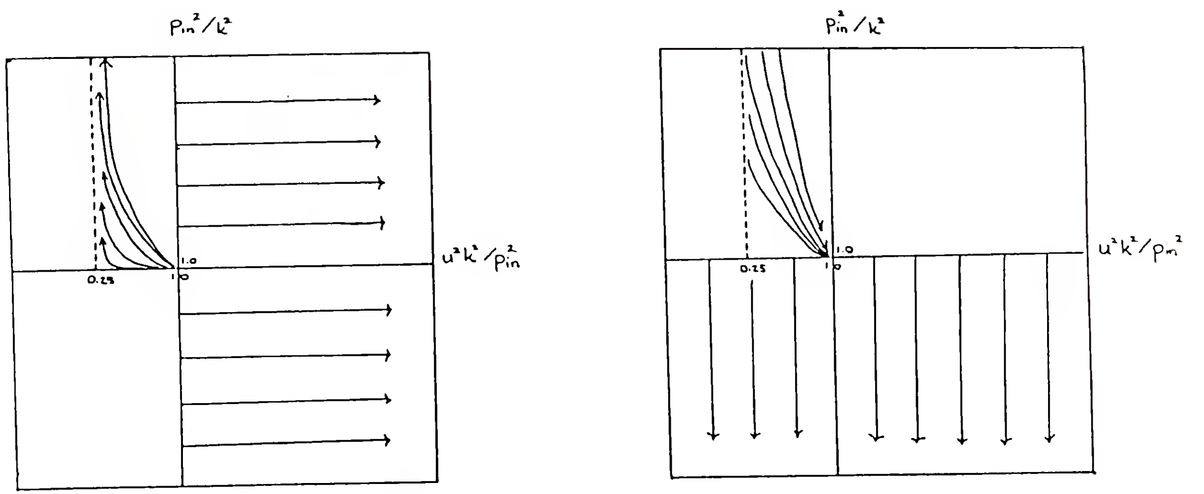

FIGURE 8-5 The Mapping of $\Gamma^{\mu}(q,p)$ Region into the $\Gamma^{\mu}(p,q)$ Region

CHAPTER IX

THE CONCLUSION

A solution has been found to the Schwinger-Dyson equations based on an approximation scheme which is characterized by the following: (1) the photon propagator is approximated by its form near the mass shell, (2) the infinite hierarchy of the vertex equation is cut off at the second order in the coupling constant and the remainder is approximated by Green's generalization of the Ward Identity for higher order diagrams.

It was found that, in order to obtain a finite solution to the electron propagator, the photon propagator had to be fixed in the Landau gauge,

$$D_{\mu\nu}(\bar{k}) = - \frac{g_{\mu\nu}}{k^2} + \frac{k_\mu k_\nu}{k^4} . \tag{9-1}$$

The bare mass of the electron was found to be zero as predicted by Johnson, Wiley and Baker.[13] Unlike Johnson, Baker and Willey, who concentrated on finding the asymptotic form of the electron propagator, a complete solution for all momenta was found. It took on a very simple form,

$$S(\bar{p}) = \frac{1}{A(p^2) - \not{p}\, B(p^2)} . \tag{9-2}$$

The functions A and B were defined in Eqs. (3-15) and (3-16).

An analytic solution to a simplified approximation of the vertex equation was found by H.S. Green. His solution forms the basis of the

Mass Shell Solution. This expression formed an extraordinarily successful solution to the complete vertex equation (approximate only to the extent of points (1) and (2) above). This analytic solution was represented by a numerical process and tested for its range of applicability. It was found to be a good representation of the solution, not only in the region for which it was designed but also in the region of an electron momentum squared as large as 10^{12}. This solution was then extended by making some deductions about how the solution should behave in the asymptotic region. The Extended Mass Shell Solution was found to be a good solution for the range of the variables,

$$p^2/k^2 > 10^{-3}$$
$$u^2k^2/p^2 < 10^5, \tag{9-3}$$

where p^2 is the incoming momentum squared of the vertex, $\Gamma^\mu(\bar{p},\bar{p}+\bar{k})$, k^2 is the photon momentum squared and $u = \bar{p}\cdot\bar{k}/(p^2k^2)^{\frac{1}{2}}$. The Extended Mass Shell Solution provides an identification of the transverse vertex function.

$$\begin{aligned}\Gamma^\mu(\bar{p},\bar{p}+\bar{k}) = {} & \frac{p^\mu}{pk}\,\overline{F}(p^2/k^2,\ u^2k^2/p^2,\ k^2/m^2) \\ & + \gamma^\mu\,\overline{G}_0(p^2/k^2,\ u^2k^2/p^2,\ k^2/m^2) \\ & + \frac{p^\mu \not{p}}{p^2}\,\overline{G}_1(p^2/k^2,\ u^2k^2/p^2,\ k^2/m^2) \\ & + p^\mu \not{k}\,\frac{\overline{G}_2}{pk}\,(p^2/k^2,\ u^2k^2/p^2,\ k^2/m^2) \\ & + p^\mu[\not{p},\not{k}]\,\frac{\overline{H}_0}{p^2k^2}\,(p^2/k^2,\ u^2k^2/p^2,\ k^2/m^2)\end{aligned}$$

$$+ [\gamma^\mu, \not{p}] \frac{\overline{H}_1}{pk} (p^2/k^2,\ u^2k^2/p^2,\ k^2/m^2)$$

$$+ [\gamma^\mu, k] \frac{\overline{H}_2}{k^2} (p^2/k^2,\ u^2k^2/p^2,\ k^2/m^2)$$

$$+ \varepsilon^{\mu\alpha\nu\phi} \gamma^5 \gamma_\phi p_\nu k_\alpha \frac{\overline{I}}{pk} (p^2/k^2,\ u^2k^2/p^2,\ k^2/m^2) \tag{9-4}$$

where the eight scalar functions $\overline{F}$, $\overline{G}_o$, $\overline{G}_1$, $\overline{G}_2$, $\overline{H}_o$, $\overline{H}_1$, $\overline{H}_2$ and $\overline{I}$ are defined in Table 8-2 and pk is defined in Eq. (8-1). The longitudinal part of the vertex is given by

$$\Gamma^\mu_{long}(\bar{p},\bar{p}+\bar{k}) = \frac{k^\mu}{k^2} [A((\bar{p}+\bar{k})^2) - A(p^2)] - \frac{k^\mu \not{q}}{k^2} + \frac{k^\mu \not{p}}{k^2} . \tag{9-5}$$

The above description of the transverse part of the vertex fails outside the region bounded by Eq. (9-3). However a symmetry principle can be used to prescribe the value of the transverse vertex at a point, $(q^2/k^2,\ u_q^2k^2/q^2,\ k^2/m^2)$, outside of the region, in terms of the eight scalar functions evaluated at the point, $(p^2/k^2,\ u^2k^2/p^2,\ k^2/m^2)$. In this case

$$\Gamma^\mu(\bar{q},\bar{p}) = \frac{p^\mu}{pk} \overline{F}(p^2/k^2,\ u^2k^2/p^2,\ k^2/m^2)$$

$$+ \gamma^\mu \overline{G}_o (p^2/k^2,\ u^2k^2/p^2,\ k^2/m^2)$$

$$+ \frac{p^\mu \not{p}}{p^2} \overline{G}_1 (p^2/k^2,\ u^2k^2/p^2,\ k^2/m^2)$$

$$+ p^\mu \not{k} \frac{\overline{G}_2}{pk} (p^2/k^2,\ u^2k^2/p^2,\ k^2m^2)$$

$$- p^\mu [\not{p}, \not{k}] \frac{\overline{H}_o}{p^2k^2} (p^2/k^2,\ u^2k^2/p^2,\ k^2/m^2)$$

$$- [\gamma^{\mu},\not{p}] \frac{\overline{H}_1}{pk} (p^2/k^2,\ u^2k^2/p^2,\ k^2/m^2)$$

$$- \varepsilon^{\xi\alpha\nu\phi}\ \gamma^5\gamma_{\phi}p_{\nu}k_{\alpha} \frac{\overline{I}}{pk} (p^2/k^2,\ u^2k^2/p^2,\ k^2/m^2) \tag{9-5}$$

where the point $(q^2/k^2,\ u_q^2k^2/q^2)$ is related to the point $(p^2/k^2, u^2k^2/p^2)$ by Eqs. (8-35) and (8-36).

This solution represents a substantial improvement over all earlier attempts to find a nonperturbative solution to the vertex equation. Work[15] by other researchers has been characterized by a dependence on renormalization techniques to keep the solution finite, or else only asymptotic forms of the eight transverse functions were sought, usually involving more drastic approximations than have been applied here. Furthermore, due to the great complexity of the equations, typically only the asymptotic forms of the $\overline{I}$, $\overline{G}_0$, $\overline{G}_1$, and $\overline{G}_2$ were solved for. Thus, the Extended Mass Shell Solution to the vertex equation is sharply contrasted with earlier solutions by its completeness.

It cannot be said that the combined solutions, described here, for the electron propagator and the vertex, constitute by any means a complete resolution of the uncertainties present in the theory of Quantum Electrodynamics. However, the success of this method, to this level of approximation, is encouraging evidence that the unreasonable infinite quantities that occur in perturbation calculations of the self-energy of the electron and the vertex part are not essential. This work has demonstrated that a consistent, noninfinite solution for the electron propagator and the vertex can be found given an approximate form for the photon propagator.

Future work would look toward solving all three of the Schwinger-Dyson equations simultaneously for the three basic functions, the vertex, and electron and photon propagators. The present work gives rise to favorable expectations that a simultaneous solution to all three equations would continue to be characterized by the absence of any divergences.

APPENDIXES

APPENDIX A

DIRAC GAMMA MATRICES: DEFINITIONS AND PRODUCT RULES

P.A.M. Dirac constructed a relativisticly covariant equation for the motion of the electron. In order to achieve this, the form of the equation needed to be invariant under a Lorentz transformation. An invariant scalar quantity is expressed as a product between a contravariant and covariant vector.

$$x^{\mu}x_{\mu} = g_{\mu\nu}\, x^{\nu}x^{\mu}. \tag{A-1}$$

The metric tensor is defined by,

$$\begin{aligned} g_{\mu\nu} &= 0 \quad \text{for } \mu \neq \nu \qquad \mu,\nu = 0,1,2,3 \\ g_{jj} &= -1 \text{ for } j = 1,3 \\ g_{oo} &= 1. \end{aligned} \tag{A-2}$$

The contravariant metric tensor is defined by the relation,

$$g^{\mu\nu} = g_{\mu\nu}. \tag{A-3}$$

The contraction of $g^{\mu\alpha}g_{\alpha\nu}$ is known as δ^{μ}_{ν}.

$$\delta^{\mu}_{\nu} = g^{\mu\alpha}g_{\alpha\nu}. \tag{A-4}$$

It follows that

$$\begin{aligned} \delta^{\mu}_{\nu} &= 0 \quad \text{for } \mu \neq \nu \\ \delta^{\nu}_{\nu} &= 1 \quad \text{for } \mu = \nu\,. \end{aligned} \tag{A-5}$$

The Dirac equation for the relativistic particle with spin needed to be linear in both time and space derivatives alike and needed to correspond to the second order Klein-Gordon equation which provided for the correct energy momentum relation for a relativistic particle. To accomplish this, a familiar representation of the Dirac equation (although not the original one), employs a set of matrices called Dirac gamma matrices.

$$\gamma^{o} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\gamma^{j} = \begin{bmatrix} 0 & \sigma^{j} \\ -\sigma^{j} & 0 \end{bmatrix} \tag{A-6}$$

where σ^j represents the Pauli spin matrices.

The Dirac gamma matrices are related to the metric tensor by an anticommutation relation,

$$\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu}. \tag{A-7}$$

All possible products of gamma matrices constructs a group of 16 linearly independent 4 x 4 matrices. These are

$$I, \gamma_{\mu} , \sigma_{\mu\nu}, \gamma_5, \gamma^5\gamma_{\mu}.$$

In this work we have used the definition of the elements,

$$\sigma_{\mu\nu} = \frac{i}{2} [\gamma^{\mu} \gamma^{\nu}] \tag{A-8}$$

$$\gamma^5 = \gamma^o\gamma^1\gamma^2\gamma^3. \tag{A-9}$$

The definition of γ^5 is not standardized throughout the literature. This designation agrees with that used by J. M. Janch and F. Rohrlich[8] but differs from that used by J. Bjorken and S. Drell[17] by an i.

In working out the expansion of Eq. (4-16), products of the sixteen matrices are taken repeatedly. A multiplication table for all possible products was constructed and appears in Table A-1. In constructing the table the following definition of the Levi Civita antisymmetric tensor was used,

$$\epsilon_{\alpha\beta\gamma\delta} = 0 \quad \text{unless } \alpha,\ \beta,\ \gamma \text{ and } \delta \text{ are all different}$$

$$\epsilon_{1234} = \begin{cases} +1 & \text{for even permutations of the indices} \\ -1 & \text{for odd permutations of the indices} \end{cases}$$

$$\epsilon^{1234} = \begin{cases} -1 & \text{for even permutations of the indices} \\ +1 & \text{for odd permutations of the indices} \end{cases} \tag{A-10}$$

The following identities have proven useful:

$$4\epsilon^{\lambda\alpha\mu\phi}\gamma^5\gamma_\phi = \{\gamma^\alpha,[\gamma^\lambda,\ \gamma^\mu]\} \tag{A-11}$$

where { } enclose the anticommutator and [ ] enclose the commutator.

$$\epsilon^{\lambda\phi\gamma\nu}\ \epsilon_{\alpha\phi\beta\xi}\ \sigma^{\beta\xi} = 2\delta^\lambda_\alpha\ \sigma^{\nu\gamma} + 2\delta^\nu_\alpha\ \lambda^{\lambda\nu} + 2g^\nu_\alpha\ \sigma^{\gamma\lambda} \tag{A-12}$$

$$\epsilon^{\lambda\phi\gamma\eta}\ \epsilon_{\eta\phi\beta\xi}\ \sigma^{\beta\xi} = 4\sigma^{\gamma\lambda}. \tag{A-13}$$

TABLE A-1 MULTIPLICATION TABLE FOR GAMMA MATRICES

	1	γ^η	$\sigma^{\eta\xi}$	$\gamma^5\gamma^\eta$	γ^5
1	1	γ^η	$\sigma^{\eta\xi}$	$\gamma^5\gamma^\eta$	γ^5
γ^μ	γ^μ	$g^{\mu\eta}$ $- i\sigma^{\mu\eta}$	$ig^{\mu\eta}\gamma^\xi - ig^{\mu\xi}\gamma^5$ $-i\varepsilon^{\mu\eta\xi\cdot}{}_\phi\gamma^5\gamma^\phi$	$-g^{\mu\eta}\gamma^5$ $-\frac{i}{2}\varepsilon^{\mu\eta\cdot\cdot}{}_{\alpha\beta}\sigma^{\alpha\beta}$	$-\gamma^5\gamma^\mu$
$\sigma^{\mu\nu}$	$\sigma^{\mu\nu}$	$ig^{\nu\eta}\gamma^\mu - ig^{\eta\mu}\gamma^\nu$ $i\varepsilon^{\mu\nu\eta\cdot}{}_\phi\gamma^5\gamma^\phi$	$ig^{\mu\eta}g^{\nu\xi} - g^{\mu\xi}g^{\nu\eta}$ $-ig^{\mu\eta}\sigma^{\nu\xi} + ig^{\nu\eta}\sigma^{\mu\xi}$ $+ig^{\mu\xi}\sigma^{\nu\eta} - ig^{\nu\xi}\sigma^{\mu\eta}$ $+\varepsilon^{\mu\nu\eta\xi}\gamma^5$	$- ig^{\mu\eta}\gamma^5\gamma^\nu$ $+ ig^{\nu\eta}\gamma^5\gamma^\nu$ $+ i\varepsilon^{\mu\nu\eta\cdot}{}_\phi\gamma^\phi$	$-\frac{1}{2}\varepsilon^{\mu\nu}_{\cdot\cdot}{}_{\alpha\beta}\sigma^{\alpha\beta}$
$\gamma^5\gamma^\mu$	$\gamma^5\gamma^\mu$	$g^{\mu\eta}\gamma^5$ $+\frac{1}{2}\varepsilon^{\mu\eta\cdot\cdot}{}_{\alpha\beta}\sigma^{\alpha\beta}$	$-ig^{\mu\xi}\gamma^5\gamma^\eta$ $+ ig^{\mu\eta}\gamma^5\gamma^\xi$ $+ i\varepsilon^{\mu\eta\xi\cdot}{}_\phi\gamma^\phi$	$- i\sigma^{\mu\eta}$ $+ g^{\mu\eta}$	$-\gamma^\mu$
γ^5	γ^5	$\gamma^5\gamma^\eta$	$-\frac{1}{2}\varepsilon^{\eta\xi\cdot\cdot}{}_{\alpha\beta}\sigma^{\alpha\beta}$	$-\gamma^\eta$	-1

where $\gamma^5 = \gamma^o\gamma^1\gamma^2\gamma^3$ and $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu \ \gamma^\nu]$

APPENDIX B

DERIVATION OF GREEN'S MASS SHELL SOLUTION

B-1 The Vertex Equation Near the Mass Shell

The equation for the vertex function is

$$\nabla^2\Gamma^\lambda(p_1,p_2) = \varepsilon(\frac{1}{2}\gamma_\nu F^{\nu\lambda} + \partial_\nu \not{p}^{-1} F^{\nu\lambda}) \tag{B-1}$$

where

$$\partial_\nu = \frac{\partial}{\partial p_\nu}, \quad \nabla = \gamma^\alpha \partial_\alpha, \quad \bar{p}_1 = \bar{p} + \bar{k}, \ \bar{p}_2 = \bar{p} \tag{B-2}$$

and

$$F^{\nu\lambda} = S(p_1)\frac{\partial}{\partial p_\nu}[\Gamma^\lambda(p_1,p_2)S(p_2)]\,S^{-1}(p_2). \tag{B-3}$$

This eauation can be modified slightly by the approximation

$$F^{\nu\lambda} \to -\ S(p_1)\Gamma^\lambda(p_1p_2)S(p_2)\gamma^\nu. \tag{B-4}$$

This will be a good approximation in the region of the mass shell where Γ^λ approaches γ^λ and $S^{-1}(p)$ is the slowly changing function, $\not{p} + A(p^2)$.

The following definition is introduced,

$$\Delta^\lambda = S(p_1)\Gamma^\lambda(p_1,p_2)S(p_2). \tag{B-5}$$

Now a solution is sought to the modified vertex equation,

$$\nabla^2\Gamma^\lambda = \varepsilon(\frac{1}{2}\gamma_\nu\Delta^\lambda\gamma^\nu + \partial_\nu \not{p}^{-1}\Delta^\lambda\gamma^\nu). \tag{B-6}$$

This equation is expected on the basis of the approximation in Eq. (B-4) to be a faithful representation of (B-1) equation near the mass shell.

The following tensors are defined

$$C^{\lambda} = \frac{1}{4}\,\mathrm{tr}\,[\Gamma^{\lambda}] \tag{B-7}$$

$$C^{\lambda}_{\mu} = \frac{1}{4}\,\mathrm{tr}\,[\Gamma^{\lambda}\gamma_{\mu}] \tag{B-8}$$

$$C^{\lambda}_{\mu\nu} = \frac{1}{4}\,\mathrm{tr}\,[\Gamma^{\lambda}\gamma_{\mu\nu}] \tag{B-9}$$

$$C_{\mu\nu\rho} = \frac{1}{4}\,\mathrm{tr}\,[\Gamma^{\lambda}\gamma_{\mu\nu\rho}] \tag{B-10}$$

$$D^{\lambda} = \frac{1}{4}\,\mathrm{tr}\,[\Delta^{\lambda}] \tag{B-11}$$

$$D^{\lambda}_{\mu} = \frac{1}{4}\,\mathrm{tr}\,[\Delta^{\lambda}\gamma_{\mu}] \tag{B-12}$$

$$D^{\lambda}_{\mu\nu} = \frac{1}{4}\,\mathrm{tr}\,[\Delta^{\lambda}\gamma_{\mu\nu\rho}] \tag{B-13}$$

$$D^{\lambda}_{\mu\nu\rho} = \frac{1}{4}\,\mathrm{tr}\,[\Delta^{\lambda}\gamma_{\mu\nu\rho}] \tag{B-14}$$

where

$$\gamma_{\mu\nu} = \frac{1}{2}\,[\gamma_{\mu},\gamma_{\nu}]$$

$$\gamma_{\mu\nu\rho} = \frac{1}{2}\,\{\gamma_{\mu},\gamma_{\nu\rho}\}\ .$$

Take the product of Eq. (B-6) with each matrix, Π, γ^{λ}, $\gamma_{\mu\nu}$ and $\gamma_{\mu\nu\rho}$ and perform the trace. This converts Eq. (A-6) into a system of four equations:

$$\nabla^2 C^{\lambda} = -\ 3\,\varepsilon\,D^{\lambda} \tag{B-15-a}$$

$$\nabla^2 C^{\lambda}_{\mu} = 2\,\varepsilon\,D^{\lambda}_{\mu} - 2\,\varepsilon\,\partial_{\mu}\nabla^{-2}\partial^{\nu}D^{\lambda}_{\nu} \tag{B-15-b}$$

$$\nabla^2 C^{\lambda}_{\mu\nu} = -\ \varepsilon\, D^{\lambda}_{\mu\nu} - 2\,\varepsilon\,\partial_{\mu}\nabla^{-2}\partial^{\rho}D^{\lambda}_{\nu\rho}$$

$$+\ 2\,\varepsilon\,\partial_{\nu}\nabla^{-2}\partial^{\rho}D^{\lambda}_{\mu\rho} \tag{B-15-c}$$

$$\nabla^2 C^{\lambda}_{\mu\nu\rho} = -\ 2\,\varepsilon\,\partial_{\mu}\nabla^{-2}\partial^{\sigma}D^{\lambda}_{\nu\rho\sigma}$$

$$-\ 2\,\varepsilon\,\partial_{\nu}\nabla^{-2}\partial^{\sigma}D^{\lambda}_{\rho\mu\sigma}$$

$$-\ 2\,\varepsilon\,\partial_{\rho}\nabla^{-2}\partial^{\sigma}D^{\lambda}_{\mu\nu\sigma}. \tag{B-15-d}$$

The electron propagator is given by

$$S(p_1) = \frac{\not{p}_1 + A_1}{p_1^2 - A_1^2} \quad \text{and} \quad S(p_2) = \frac{\not{p}_2 + A_2}{p_2^2 - A_2^2} \tag{B-16}$$

where

$$\bar{p}_1 = \bar{p} + \bar{k}, \quad \bar{p}_2 = \bar{p}, \quad A(p_1^2) = A_1, \quad A(p_2^2) = A_2$$

and the A function is the electron propagator function which was determined in Chapter III. Allow the abbreviation $D_{12} = (p_1^2-A_1^2)(p_2^2-A_2^2)$. When this is done the D^{λ} tensors are expressed as

$$D^{\lambda} = \frac{1}{4}\,\mathrm{tr}\,[\Gamma^{\lambda}(\not{p}_2+A_2)(\not{p}_1+A_1)]/D_{12}$$

$$= [(p_1 \circ p_2 + A_1A_2)C^{\lambda} + (A_1 p_2^{\mu} + A_2 p_i^{\mu})C^{\lambda}_{\mu}$$

$$-\ p_1^{\mu}p_2^{\nu}C^{\lambda}_{\mu\nu}]/D_{12} \tag{B-17-a}$$

$$D^{\lambda}_{\mu} = \frac{1}{4}\,\mathrm{tr}\,[\Gamma^{\lambda}(\not{p}_2+A_2)\gamma_{\mu}(\not{p}_1+A_1)]/D_{12}$$

$$= [(A_2 p_{1_\mu} + A_1 p_{2_\mu})C^{\lambda} + (A_1A_2 - p_1\cdot p_2)C^{\lambda}_{\mu}$$

$$+\ (p_{1_\mu}p_2^{\nu} + p_{2_\mu}p_1^{\nu})C^{\lambda}_{\nu} + (A_2p_1^{\nu}-A_1p_2^{\nu})C^{\lambda}_{\mu\nu} + p_1^{\nu}p_2^{\rho}C^{\lambda}_{\mu\nu\rho}]/D12 \tag{B-17-b}$$

$$D^{\lambda}_{\mu\nu} = \frac{1}{4}\,\mathrm{tr}\,[\Gamma^{\lambda}(\not{p}_2+A_2)\gamma_{\mu\nu}(\not{p}_1+A_1)]/D_{12}$$

$$= [-\,(p_{1\mu}p_{2\nu}-p_{2\mu}p_{1\nu})C^{\lambda} + (A_2p_{1\nu}-A_1p_{2\nu})C^{\lambda}_{\mu}$$

$$-\,(A_2p_{1\mu} - A_1p_{2\mu})C^{\lambda}_{\nu} + (p_1\cdot p_2 + A_1A_2)C^{\lambda}_{\mu\nu}$$

$$-\,(p_{1\nu}p_2^{\rho}+p_{2\nu}p_1^{\rho})C^{\lambda}_{\mu\rho} + (p_{1\mu}p_2^{\rho} + p_{2\mu}p_1^{\rho})C^{\lambda}_{\nu\rho}$$

$$+\,(A_2p_1^{\rho} + A_1p_2^{\rho})C^{\lambda}_{\mu\nu\rho}]/D_{12} \qquad \text{(B-17-c)}$$

$$D^{\lambda}_{\mu\nu\rho} = \frac{1}{4}\,\mathrm{tr}\,[\Gamma^{\lambda}(\not{p}_2-A_2)\gamma_{\mu\nu\rho}(\not{p}_1+A_1)]/D_{12}$$

$$= [(p_{1\mu}p_{2\nu} - p_{2\mu}p_{1\nu})C^{\lambda}_{\rho} + (p_{1\nu}p_{2\rho} - p_2\;p_1\;)C^{\lambda}_{\mu}$$

$$+\,(p_{1\rho}p_{2\mu}-p_{2\rho}p_{1\mu})C^{\lambda}_{\nu} + (A_2p_{1\mu}+A_1p_{2\mu})C^{\lambda}_{\nu\rho}$$

$$+\,(A_2p_{1\nu}+A_1p_{2\nu})C^{\lambda}_{\rho\mu} + (A_2p_{1\rho}+A_1p_{2\rho})C^{\lambda}_{\mu\nu}$$

$$+\,(A_1A_2-p_1\cdot p_2)C^{\lambda}_{\mu\nu\rho} + (p_{1\mu}p_2^{\sigma}+p_{2\mu}p_1^{\sigma})C^{\lambda}_{\nu\rho\sigma}$$

$$+\,(p_{1\nu}p_2^{\sigma}+p_{2\nu}p_1^{\sigma})C^{\lambda}_{\rho\mu\sigma} + (p_{1\rho}p_2^{\sigma}+p_{2\rho}p_1^{\sigma})C^{\lambda}_{\mu\nu\sigma} \qquad \text{(B-17-d)}$$

The C^{λ} tensors can be expressed in terms of the eight scalar functions of the transverse vertex, where $\Gamma^{\lambda}_{\mathrm{trans}}$ is given by

$$\Gamma^{\lambda}_{\mathrm{trans}} = p^{\lambda}\frac{F}{kp} + p^{\lambda}[\not{p},\not{k}]\frac{H_o}{k^2p^2} + [\gamma^{\lambda},\not{p}]\frac{H_1}{kp} + [\gamma^{\lambda},\not{k}]\frac{H_2}{k^2}$$

$$+\,\gamma^{\lambda}G_o + p^{\lambda}\not{p}\frac{G_1}{p^2} + p^{\lambda}\not{k}\frac{G_2}{kp} + \varepsilon^{\lambda\alpha\nu\phi}\gamma^5\gamma_{\phi}k_{\alpha}p_{\nu}\frac{I}{kp} \qquad \text{(B-18)}$$

the associated tensors are

$$C^{\lambda} = p^{\lambda} \frac{F}{kp} \tag{B-19-a}$$

$$C^{\lambda}_{\mu} = \delta^{\lambda}_{\mu} G_{o} + p^{\lambda} p_{\mu} \frac{G_1}{p^2} + p^{\lambda} k_{\mu} \frac{G_2}{pk} \tag{B-19-b}$$

$$C^{\lambda}_{\mu\nu} = (p^{\lambda} p_{\nu} k_{\mu} - p^{\lambda} p_{\mu} k_{\nu}) \frac{2\ H_o}{k^2 p^2} + (\delta^{\lambda}_{\nu} k_{\mu} - \delta^{\lambda}_{\mu} k_{\nu})\ 2 \frac{H_1}{k^2}$$

$$+ (\delta^{\lambda}_{\nu} p_{\mu} - \delta^{\lambda}_{\mu} p_{\nu}) \frac{2\ H_2}{kp} \tag{B-19-c}$$

$$C^{\lambda}_{\mu\nu\rho} = [\delta^{\lambda}_{\mu}(k_{\rho}p_{\nu} - k_{\nu}p_{\rho}) + \delta^{\lambda}_{\nu}(k_{\mu}p_{\rho} - k_{\rho}p_{\mu})$$

$$+ \delta^{\lambda}_{\rho}(k_{\nu}p_{\mu} - k_{\mu}p_{\nu})] \frac{I}{kp}\ . \tag{B-19-d}$$

The system of equations contained in (B-15) is still a heavy challenge to analyze. An additional simplification is desirable. It will be assumed that the G_o term in Γ^{λ} is dominant and has its near the mass shell value of 1, and that the functions A_1 and A_2 will be closely approximated by their on the mass shall values of m_e, the experimental electron mass. These are exactly the assumptions which were made in approximating $F^{\lambda\nu}$ by $\Delta^{\lambda}\gamma^{\nu}$ in Eq. (B-6). No new restrictions to the solution have been made. The effect of G_o being dominant and equal to 1 is C^{λ}_{μ} becomes δ^{λ}_{μ} and the contributions of C^{λ}, $C^{\lambda}_{\mu\nu}$ and $C^{\lambda}_{\mu\nu\rho}$ are small relative to C^{λ}_{μ} and can be neglected in Eq. (B-17). The new simplified version of (B-17) is

$$D^{\lambda} = 2\ m\ p^{\lambda}/D_{12} \tag{B-20-a}$$

$$D^{\lambda}_{\mu} = [(m^2 - p_1 \cdot p_2)\delta^{\lambda}_{\mu} + (p_{1\mu} + p_{2\mu})p^{\lambda}]/D_{12} \tag{B-20-b}$$

$$D^{\lambda}_{\mu\nu} = m[(p_{1\nu}-p_{2\nu})\delta^{\lambda}_{\mu} - (p_{1\mu}-p_{2\mu})\delta^{\lambda}_{\nu}]/D_{12} \tag{B-20-c}$$

$$D^{\lambda}_{\mu\nu\rho} = [(p_{1\mu}p_{2\nu}-p_{2\mu}p_{1\nu})\delta^{\lambda}_{\rho} + (p_{1\nu}p_{2\rho}-p_{2\nu}p_{1\rho})\delta^{\lambda}_{\mu}$$

$$+ (p_{1\rho}p_{2\mu}-p_{2\rho}p_{1\mu})\delta^{\lambda}_{\nu}]/D_{12}. \tag{B-20-d}$$

The new expressions for the D^{λ} tensors will be used to solve the four equations in (B-15). To facilitate the solution observe the following general relationships.

Let $x_a = (x+a)^2 = x^2 + 2xa + a^2$.

If

$$\nabla^2\phi\left(\frac{x_a}{m^2}\right) = \frac{1}{x_a-m^2} \tag{B-21}$$

then

$$\phi\left(\frac{x_a}{m^2}\right) = \frac{1}{4}\left(1 - \frac{m^2}{x_a}\right) \ln \left(1 - \frac{x_a}{m^2}\right). \tag{B-22}$$

Let

$$x_\beta = [\bar{p} + (\beta+1)\tfrac{1}{2}\ \bar{k}]^2 \text{ and } u_\beta = m^2 - \tfrac{1}{4}(1-\beta^2)k^2. \tag{B-23}$$

If

$$\nabla^2\Phi_A = \frac{1}{(p_1^2-m^2)(p_2^2-m^2)}$$

$$= \frac{1}{2}\int_{-1}^{1} \frac{d\beta}{(x_\beta-u_\beta)^2}, \tag{B-24}$$

then

$$\Phi_A = -\frac{1}{8}\int_{-1}^{1} \frac{d\beta}{x_\beta} \ln \left(1 - \frac{x_\beta}{u_\beta}\right). \tag{B-25}$$

If

$$\nabla^2\Phi_B = \frac{\ln \dfrac{p_1^2-m^2}{p_2^2-m^2}}{p_1^2 - p_2^2} = \frac{1}{2}\int_{-1}^{1} \frac{d\beta}{x_\beta-u_\beta}, \tag{B-26}$$

then

$$\Phi_B = \frac{1}{2}\int_{-1}^{1} d\beta\ \Phi(\frac{x_\beta}{u_\beta})$$

$$= \frac{1}{8}\int_{-1}^{1} d\beta(1 - \frac{u_\beta}{x_\beta})\ \ln\ (1 - \frac{x_\beta}{u_\beta}). \tag{B-27}$$

If $$\nabla^2\Phi_c = \int_{-1}^{1} \frac{d\beta}{x\beta}\ (m^2 - u_\beta)\ \ln\ (1 - \frac{x_\beta}{u_\beta})\ , \tag{B-28}$$

then

$$\Phi_c = \frac{1}{4}\int_{-1}^{1} d\beta(m^2 - u_\beta)\left[L_2(\frac{x_\beta}{u_\beta}) - (1 - \frac{u_\beta}{x_\beta})\ \ln\ (1 - \frac{x_\beta}{u_\beta})\right] \tag{B-29}$$

where

$$L_2(z) = -\int_0^z \ln\ (1-z)\frac{dz}{z} \tag{B-30}$$

so that

$$\int_0^z L_2(z)dz = z\ L_2(z) + (z-1)\ \ln\ (1-z) - z.$$

Using these general transformation rules, the system of Eqs. (13-15) can be solved.

B-2 Solution to Equation (B-15-a)

Substituting the definition of D^λ in (B-20-a) into Eq. (B-15-a) yields

$$\nabla^2 C^\lambda = \frac{-\ 6\varepsilon m p^\lambda}{(p_1^2 - m^2)(p_2^2 - m^2)}$$

$$= 3\varepsilon m \partial^\lambda \left[\frac{\ln \dfrac{p_1^2 - m^2}{p_2^2 - m^2}}{p_1^2 - p_2^2}\right]. \tag{B-31}$$

By using the definition of Φ_B in (B-27) and letting

$\Phi'(z) = \frac{\partial}{\partial z}\,\phi(z)$, C^λ can be determined.

$$C^\lambda = 3\varepsilon m\, \partial^\lambda \Phi_B$$

$$= \frac{3}{4}\,\varepsilon m p^\lambda \int_{-1}^{1} \frac{d_\beta}{u_\beta}\;\phi'\left(\frac{x_\beta}{u\beta}\right)$$

$$= \frac{3}{4}\,\varepsilon m p^\lambda \int_{-1}^{1} \frac{d_\beta}{x_\beta}\left[\frac{u_\beta}{x_\beta}\ln\left(1\frac{x_\beta}{u_\beta}\right) + 1\right] . \tag{B-32}$$

This equation is used in Chapter VI to evaluate the scalar function $F(p^2,\mu)$ through the relationship provided by (B-19-a).

B-3 Solving Equation B-15-b

Substituting the definition of D^λ_μ in (B-20-b) into Eq. (B-15-b) yields

$$\nabla^2 C^\lambda_\mu = 2\varepsilon\left[\frac{(m^2-p_1 p_2)\delta^\lambda_\mu + (p_{1\mu}+p_{2\mu})p^\lambda - \partial_\mu E^\lambda}{(p_1^2-m^2)(p_2^2-m^2)}\right] \tag{B-33}$$

where

$$\nabla^2 E^\lambda = \partial^\nu\left[\frac{(m^2-p_1 p_2)\delta^\lambda_\nu + (p_{1\nu}+p_{2\nu})p\lambda}{(p_1^2-m^2)(p_2^2-m^2)}\right] . \tag{B-34}$$

Let $C^\lambda_\mu = C^*\delta^\lambda_\mu - \partial^\lambda C^*_\mu$. (B-35)

Since

$$\frac{(m^2-p_1\cdot p_2)\delta^\lambda_\mu + (p_{1\mu}+p_{2\mu})p^\lambda}{(p_1^2-m^2)(p_2^2-m^2)} = -\frac{1}{2}\frac{\partial}{\partial p_\lambda}\left[\frac{(p_{1\mu}+p_{2\mu})}{(p_1^2-p_2^2)}\ln\left(\frac{p_1^2-m^2}{p_2^2-m^2}\right)\right]$$

$$+ \delta^{\lambda}_{\mu}\left[\frac{m^2-p_1\cdot p_2}{(p_1^2-m^2)(p_2^2-m^2)} + \frac{1}{p_1^2-p_2^2} \ln \frac{(p_1^2-m^2)}{(p_2^2-m^2)}\right], \tag{B-36}$$

Eq. (B-33) can be reexpressed in terms of two equations.

$$\nabla^2 C^*_\mu = 2\varepsilon \frac{1}{2}\left[\frac{(p_{1\mu}+p_{2\mu})}{p_1^2-p_2^2} \ln\left(\frac{p_1^2-m^2}{p_2^2-m^2}\right) + \partial_\mu E\right] \tag{B-37}$$

and

$$\nabla^2 C^* = 2\varepsilon\left[\frac{m^2-p_1.p_2}{(p_1^2-m^2)(p_1^2-m^2)} + \frac{1}{p_1^2-p_2^2} \ln\left(\frac{p_1^2-m^2}{p_2^2-m^2}\right)\right]$$

$$= 2\varepsilon\left[-\frac{1}{2}\frac{1}{(p_1^2-m^2)} - \frac{1}{2}\frac{1}{(p_2^2-m^2)} + \frac{k^2/2}{(p_1^2-m^2)(p_2^2-m^2)}\right.$$

$$\left. + \frac{1}{(p_1^2-p_2^2)} \ln\left(\frac{p_1^2-m^2}{p_2^2-m^2}\right)\right] \tag{B-38}$$

with

$$\nabla^2 E = \frac{m^2-p_1\, p_2}{(p_1^2-m^2)(p_2^2-m^2)} + \frac{1}{(p_1^2-p_2^2)} \ln\left(\frac{p_1^2-m^2}{p_2^2-m^2}\right)$$

$$- \frac{1}{2}\partial^\mu\left[\frac{(p_1\ \ +p_2\ \ )}{(p_1^2-p_2^2)} \ln\left(\frac{p_1^2-m^2}{p_2^2-m^2}\right)\right]$$

$$= \frac{2m^2}{(p_1^2-m^2)(p_2^2-m^2)} - \frac{2}{(p_1^2-p_2^2)} \ln\left(\frac{p_1^2-m^2}{p_2^2-m^2}\right) . \tag{B-39}$$

From (B-38) and using the definitions of Φ, Φ_A and Φ_B it can be seen that

$$C^* = -\ \varepsilon\Phi\left(\frac{p_1^2}{m^2}\right) - \varepsilon\Phi\left(\frac{p_2^2}{m^2}\right) + \varepsilon k^2\Phi_A + 2\varepsilon\Phi_B$$

$$= \frac{\varepsilon}{4}\left\{\int_{-1}^{1}\left[(1-\frac{u_\beta}{x_\beta}-\frac{k^2}{2x_\beta^2})\ \ln\ (1-\frac{x_\beta}{u_\beta})\right] d\beta\right.$$

$$\left. -\ (1-\frac{m^2}{p_1^2})\ \ln\left[(1-\frac{p_1^2}{m^2}) - (1-\frac{m^2}{p_2^2})\ \ln\ (1-\frac{p_2^2}{m^2})\right]\right\}. \qquad \text{(B-40)}$$

From (B-37) an expression for the transverse and longitudinal components of C^*_μ, $\tilde{C}^*_\mu$ and $k_\mu C^*_\alpha k^\alpha/k^2$, can be derived

$$\nabla^2\tilde{C}^*_\mu = 2\varepsilon\left[\frac{\tilde{p}_\mu}{(p_1^2-p_2^2)}\ \ln\left(\frac{p_1^2-m^2}{p_2^2-m^2}\right) + \tilde{\partial}_\mu E\right]$$

$$= 2\varepsilon\left[\frac{1}{2}\ \tilde{p}_\mu \int_{-1}^{1} \frac{d\beta}{x_\beta - u_\beta} + \tilde{\partial}_\mu E\right]$$

$$= 2\varepsilon\left[\tilde{\partial}_\mu\ \frac{1}{4}\int_{-1}^{1} d\beta\ \ln\ (1-\frac{x_\beta}{u_\beta}) + E\right]$$

$$= -\ 2\varepsilon\ \tilde{\partial}_\mu\left[\frac{1}{4}\int_{-1}^{1}\frac{d\beta}{x_\beta}\ (m^2-u\beta)\ \ln\ (1-\frac{x_\beta}{u_\beta})\right] \qquad \text{(B-41)}$$

since

$$E = 2m^2\ \Phi_A - 2\Phi_B$$

$$= -\ \frac{1}{4}\int_{-1}^{1}\left[\frac{1}{x_\beta}\ (x_\beta + m^2 - u_\beta)\ \ln\ (1-\frac{x_\beta}{u_\beta})\right] d\beta\ . \qquad \text{(B-42)}$$

Using the Φ_c transformation in (B-41) it is found that

$$\tilde{C}^*_\mu = -\ \frac{1}{2}\ \varepsilon\ \tilde{\partial}_\mu \Phi_c$$

$$= \frac{1}{8}\ \varepsilon\ \tilde{\partial}_\mu \int_{-1}^{1}\left\{(m^2-u_\beta)\ \left[L_2(\frac{x_\beta}{u_\beta}) + (1-\frac{u_\beta}{x_\beta})\ \ln\ (1-\frac{x_\beta}{u_\beta})\right]\right\} d\beta$$

$$= \frac{1}{4}\ \varepsilon\ \tilde{p}_\mu \int_{-1}^{1}\frac{1}{u_\beta}\ (m^2-u_\beta)\ \left[(\frac{u_\beta}{x_\beta}-1)\ \ln\ (1-\frac{x_\beta}{u_\beta}) + 1\right] d\beta\ . \qquad \text{(B-43)}$$

The expression for the longitudinal part of C^*_μ in Eq. (B-37) yields

$$\nabla^2 k^\mu C^*_\mu = 4\varepsilon\left[\frac{1}{4}\ln\left(\frac{p_1^2-m^2}{p_2^2-m^2}\right) + \frac{k_\alpha}{2}\frac{\partial}{\partial p_\alpha} \quad E\right]. \tag{B-44}$$

Rearranging this expression,

$$\nabla^2[k^\lambda C^*_\mu - \varepsilon\,(k\cdot p + \frac{k^2}{2})\;E] = 4\varepsilon\left[\frac{1}{4}\ln\left(\frac{p_1^2-m^2}{p_2^2-m^2}\right)\right.$$

$$\left. - \frac{1}{2}\,(k\cdot p + \frac{k^2}{2})\,\nabla^2 E\right]$$

$$= 4\varepsilon\left[\frac{1}{2}\ln\left(\frac{p_1^2-m^2}{p_2^2-m^2}\right) - \frac{1}{4}\,\frac{m^2(p_1^2-p_2^2)}{(p_1^2-m^2)(p_2^2-m^2)}\right]. \tag{B-45}$$

Using the expression (B-39) for $\nabla^2 E$ it can be seen that

$$k^\alpha C^*_\alpha = \varepsilon(k\cdot p - \frac{k^2}{2})E + \frac{\varepsilon}{4}\left[(p_1^2-m^2)\ln\,(1-\frac{p_1^2}{m^2})\right.$$

$$\left. - (p_2^2-m^2)\ln\,(1-\frac{p_2^2}{m^2}) - (p_1^2-p_2^2)\right]$$

$$= -\frac{1}{4}\varepsilon(k\cdot p - \frac{k^2}{2}\int_{-1}^{1}\frac{(m^2-u_\beta)}{x_\beta}\ln\,(1-\frac{x_\beta}{u_\beta})d\beta. \tag{B-46}$$

By combining the longitudinal part with the transverse part C^*_μ is defined

$$C^*_\mu = \frac{1}{4}\varepsilon\,\tilde{p}_\mu\int_{-1}^{1}\frac{1}{x_\beta}\,(m^2-u_\beta)\left[\frac{(u_\beta-1)}{x_\beta}\ln\,(1-\frac{x_\beta}{u_\beta}) + 1\right]d\beta$$

$$-\frac{\varepsilon}{4}\,k_\mu(\frac{k\cdot p}{k^2} + \frac{1}{2})\int_{-1}^{1}\frac{1}{x_\beta}\,(m^2-u_\beta)\ln\,(1\frac{x_\beta}{u_\beta})\,d\beta\quad . \tag{B-47}$$

Therefore (B-40) and (B-47) provide the needed information to define C_μ^λ since

$$C_\mu^\lambda = C^* \delta_\mu^\lambda - \partial^\lambda C_\mu^* . \qquad \text{(B-48)}$$

Equations (B-40), (B-47) and (B-48) are used in Chapter VI to evaluate the scalar functions G_o, G_1 and G_2 through the relationship provided by (B-19-b).

B-4 Solving Equation (B-15-c)

To find the definition of the $C_{\mu\nu}^\lambda$ it is necessary to solve Eq. (B-15-c),

$$\nabla^2 C_{\mu\nu}^\lambda = -\varepsilon D_{\mu\nu}^\lambda - 2\varepsilon \partial_\mu \nabla^{-2} \partial^\rho D_{\nu\rho}^\lambda + 2\varepsilon \partial_\nu \nabla^{-2} \partial^\rho D_{\mu\rho}^\lambda . \qquad \text{(B-15-c)}$$

Alternatively, since

$$\nabla^2 \partial^\nu C_{\mu\nu}^\lambda = \varepsilon \partial^\nu D_{\mu\nu}^\lambda \qquad \text{(B-49)}$$

so that

$$\varepsilon \nabla^{-2} \partial^\nu D_{\mu\nu}^\lambda = \partial^\nu C_{\mu\nu}^\lambda \qquad \text{(B-50)}$$

Eq. (B-15-c) could be equivalently expressed as

$$\nabla^2 C_{\mu\nu}^\lambda + 2\partial_\mu \partial^\rho C_{\nu\rho}^\lambda - 2\partial_\nu \partial^\rho C_{\mu\rho}^\lambda = -\varepsilon D_{\mu\nu}^\lambda . \qquad \text{(B-51)}$$

Inserting the definition of $D_{\mu\nu}^\lambda$ given in (B-20-c) into this yields

$$\nabla^2 \partial^\nu C_{\mu\nu}^\lambda = \varepsilon m(\delta_\mu^\lambda k_\alpha \partial^\alpha - k_\mu \partial^\lambda) \frac{1}{(p_1^2 - m^2)(p_2^2 - m^2)} . \qquad \text{(B-52)}$$

From the Φ_A relationship it can be seen that

$$\partial^\nu C^\lambda_{\mu\nu} = \epsilon m(\delta^\lambda_\mu\, k_\alpha \partial^\alpha - k_\mu \partial^\lambda)\Phi_A$$

$$= -\frac{1}{8}\, \epsilon m(\delta^\lambda_\mu k_\alpha \partial^\alpha - k_\mu \partial^\lambda) \int_{-1}^{1} \frac{d\beta}{x_\beta} \ln(1 - \frac{x_\beta}{u_\beta}). \qquad \text{(B-53)}$$

This can be used to form the terms $\partial_\nu \partial^\rho C^\lambda_{\mu\rho}$ and $\partial_\mu \partial^\rho C^\lambda_{\nu\rho}$ in Eq. (B-50). Equation (B-50) then becomes

$$\nabla^2 C^\lambda_{\mu\nu} = \epsilon m[-(\delta^\lambda_\mu k_\alpha \partial^\alpha - k_\mu \partial^\lambda)\partial_\nu$$
$$+ (\delta^\lambda_\nu k_\alpha \partial^\alpha - k_\nu \partial^\lambda)\partial_\mu] \int_{-1}^{1} \frac{1}{x_\beta} \ln(1 - \frac{x_\beta}{u_\beta}) d\beta$$
$$- \epsilon m(\delta^\lambda_\mu k_\nu - \delta^\lambda_\nu k_\mu) \frac{1}{(p_1^2 - m^2)(p_2^2 - m^2)}. \qquad \text{(B-54)}$$

Using the transformation relationships for Φ_A and Φ_C it is possible to show that

$$C^\lambda_{\mu\nu} = \frac{1}{16}\, \epsilon m[(\delta^\lambda_\mu k_\alpha \partial^\alpha - k_\mu \partial^\lambda)\partial_\nu - (\delta^\lambda_\nu k_\alpha \partial^\alpha - k_\nu \partial^\lambda \partial_\mu)]$$
$$\int_{-1}^{1} [L_2(\frac{x_\beta}{u_\beta}) - (1 - \frac{u_\beta}{x_\beta}) \ln(1 - \frac{x_\beta}{u_\beta})] d\beta$$
$$+ \frac{1}{8}\, \epsilon m(\delta^\lambda_\mu k_\nu - \delta^\lambda_\nu k_\mu) \int_{-1}^{1} \ln(1 - \frac{x_\beta}{u_\beta}) \frac{1}{x_\beta} d\beta. \qquad \text{(B-55)}$$

Equation (B-55) is used in Chapter VI to evaluate the scalar functions H_o, H_1 and H_2 through the relationship provided by (B-19-c).

B-5 Solving Equation (B-15-d)

The last equation, (B-15-d), can be put into a simpler form by performing a differentiation with respect to p^{ρ}. This yields

$$\nabla^2 \partial^{\rho} C^{\lambda}_{\mu\nu\rho} = - 2\varepsilon \partial^{\rho} D^{\lambda}_{\mu\nu\rho} . \tag{B-56}$$

Using the definition of $D^{\lambda}_{\mu\nu\rho}$ in Eq. (B-20-d) this becomes

$$\nabla^2 \partial^{\rho} C^{\lambda}_{\mu\nu\rho} = - 2\varepsilon \left\{ \partial^{\lambda} \frac{(p_{1\mu} p_{2\nu} - p_{2\mu} p_{1\nu})}{(p_1^2 - m^2)(p_2^2 - m^2)} + \partial^{\rho} \left[\frac{(p_{1\nu} p_{2\rho} - p_{2\nu} p_{1\rho}) \delta^{\lambda}_{\mu} + (p_{1\rho} p_{2\mu} - p_{2\rho} p_{1\mu}) \delta^{\lambda}_{\nu}}{(p_1^2 - m^2)(p_2^2 - m^2)} \right] \right\} . \tag{B-57}$$

When Eq. (B-56) is multiplied by $p_1^{\mu} p_2^{\nu}$ a new equation is formed,

$$\nabla^2 p_1^{\lambda} p_2^{\nu} \partial^{\rho} C^{\lambda}_{\mu\nu\rho} = - 2\varepsilon \left[\partial^{\lambda} \left(\frac{k^2 p^2}{(p_1^2 - m^2)(p_2^2 - m^2)} - \partial^{\rho} \frac{(k^2 p^{\lambda} p_{\rho}}{(p_1^2 - m^2)(p_2^2 - m^2)} \right) \right]$$

$$= \frac{4\varepsilon k^2 p^{\lambda}}{(p_1^2 - m^2)(p_2^2 - m^2)} . \tag{B-58}$$

In the above statement use was made of the fact that

$$p_1^{\mu} p_2^{\nu} \nabla^2 \partial^{\rho} C^{\lambda}_{\mu\nu\rho} = \nabla^2 p_1^{\mu} \; p_2^{\nu} \partial^{\rho} C^{\lambda}_{\mu\nu\rho} ,$$

a consequence of the antisymmetry of $C^{\lambda}_{\mu\nu\rho}$. Using the Φ_A transform again gives

$$\nabla^2 (p_1^{\mu} p_2^{\nu} \partial^{\rho} C^{\lambda}_{\mu\nu\rho}) = \varepsilon k^2 \partial^{\lambda} \int \frac{d\beta}{x_{\beta} - u_{\beta}} . \tag{B-59}$$

Equation (B-59) is used in Chapter VI to evaluate the scalar function I through the relationship provided by (B-19-d).

APPENDIX C

FORTRAN PROGRAMS

Presented on the following pages are the Fortran programs which were used to code the Main Program and the Extended Mass Shell Program.

```
      IMPLICIT REAL*8 (A-H,O-Z)
C THIS IS THE MAIN PROGRAM
C THE EIGHT DIFFERENCIAL EQUATIONS ARE DEFINED FOR THE VERTEX EQUATION
C THIS PROGRAM READS THE VALUE OF P2,U,K2,SKL,SIGN ( ELECTRON MOMENTUM
C SQUARED, ANGLE VARIABLE, PHOTON MOMENTUM SQUARED, THE SCALE AND THE
C SIGN OF P2 AND K2, BOTH MUST HAVE THE SAME SIGN.)
      DIMENSION DX2(8)
      COMMON/SIGNPK/SIGN
      COMMON/SCALE/SKL
      WRITE(6,701)
 701  FORMAT(' NI,IX FOR RESULTS,FUNCTIONS,STORAGE')
      READ(9,*) NI,IX
C THE NI = 1 GIVES JUST THE RESULTS OF THE EQUATIONS
C THE NI =2 GIVE RESULTS PLUS FUNCTIONS
C IX = 1 IS DEFAULT
C IX = 2 PUTS THE RESULTS PLUS FUNCTIONS INTO STOREPRINT
      READ(2,122) P2,U,AK2,SKL,SIGN
 122   FORMAT(D24.16,4D11.3)
      WRITE(6,74) P2,U,AK2,SKL
  74  FORMAT(' THE POINT IS P2,U,K2;',3D15.7/' THE SCALE',D15.7)
      WRITE(4,74) P2,U,AK2,SKL
      SM=1.D0*SKL
C      WRITE(6,17) P2,U,AK2,SM
      IF(IX.EQ.2)WRITE(4,17) P2,U,AK2
      AM2=SKL*SKL
      IF(IX.EQ.2)WRITE(10,17) P2,U,AK2,AM2
  17  FORMAT(' AT THE POINT P2,U,K2,M2',D12.4,3D12.4)
      X=FTBM(P2,U,AK2,SM,NI,IX)
C      IF(NI.EQ.2)WRITE(6,21) X
  21  FORMAT (' THE FTBM EQUALS',D20.7)
      STOP
      END
C THIS SUBROUTINE TAKES THE VALUE OF THE EIGHT FUNCTIONS AND ALL
C DERIVATIVES FROM THE DATA MATRIX (THE EXTENDED MASS SHELL SOLUTION)
C FORMAT STATEMENT 113
C AND CONTRUCTS THE LEFT AND RIGHT HAND SIDES.
      FUNCTION FTBM(P2,U,K2,SM,NI,IX)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION D2(8),TM(8),X(20)
      DATA EP/2.322819D-3/
      COMMON/SIGNPK/SIGN
      COMMON/SCALE/SKL
      REAL*8 K2,LPL,LSALP,LPLSN,LSLN,LGL,LPLG,LPP,LE
      DIMENSION EROR(10,5),E(8)
      COMMON/SIMP/LPL,LSALP,LPLSN,LSLN,LGL,LPLG,LPP,LE,
     C            RPL,RSALP,RPLSN,RSLN,RGL,RPLG,RPP,RE
      COMMON/SUBTT/TE(14),TES(14),TEP(14),A,AP,APP,B,BP,TO(14),TOS(14),
     C      TOP(14)
      COMMON/FUNS/XF(8,10,1,1)
      DIMENSION XM(3,10)
      I=1
      J=1
      DO 13 IF=1,8
  13  READ(2,113) (XF(IF,IT,1,1),IT=1,10)
```

```
113 FORMAT(3D24.16/3D24.16/3D24.16/D24.16)
    PK=SIGN*DSQRT(P2*K2)
    POK=DSQRT(P2/K2)
    AKOP=DSQRT(K2/P2)
    R=P2
    S=K2
    PKU=PK*U
    U2=U*U
    TU=1.D0-U2
    P4=R*R
    P6=P4*P2
    CALL TT(R,S,U,I,J)
    F=XF(1,1,I,J)
    FP=XF(1,2,I,J)
    FPP=XF(1,3,I,J)
    FPPP=XF(1,4,I,J)
    FS=XF(1,5,I,J)
    FSS=XF(1,6,I,J)
    FSSS=XF(1,7,I,J)
    FSP=XF(1,8,I,J)
    FSSP=XF(1,9,I,J)
    FSPP=XF(1,10,I,J)
    G0=XF(2,1,I,J)
    G0P=XF(2,2,I,J)
    G0PP=XF(2,3,I,J)
    G0PPP=XF(2,4,I,J)
    G0S=XF(2,5,I,J)
    G0SS=XF(2,6,I,J)
    G0SSS=XF(2,7,I,J)
    G0SP=XF(2,8,I,J)
    G0SSP=XF(2,9,I,J)
    G0SPP=XF(2,10,I,J)
    G1=XF(3,1,I,J)
    G1P=XF(3,2,I,J)
    G1PP=XF(3,3,I,J)
    G1PPP=XF(3,4,I,J)
    G1S=XF(3,5,I,J)
    G1SS=XF(3,6,I,J)
    G1SSS=XF(3,7,I,J)
    G1SP=XF(3,8,I,J)
    G1SSP=XF(3,9,I,J)
    G1SPP=XF(3,10,I,J)
    G2=XF(4,1,I,J)
    G2P=XF(4,2,I,J)
    G2PP=XF(4,3,I,J)
    G2PPP=XF(4,4,I,J)
    G2S=XF(4,5,I,J)
    G2SS=XF(4,6,I,J)
    G2SSS=XF(4,7,I,J)
    G2SP=XF(4,8,I,J)
    G2SSP=XF(4,9,I,J)
    G2SPP=XF(4,10,I,J)
    H0=XF(5,1,I,J)
    H0P=XF(5,2,I,J)
```

```
      HOPP=XF(5,3,I,J)
      HOPPP=XF(5,4,I,J)
      HOS=XF(5,5,I,J)
      HOSS=XF(5,6,I,J)
      HOSSS=XF(5,7,I,J)
      HOSP=XF(5,8,I,J)
      HOSSP=XF(5,9,I,J)
      HOSPP=XF(5,10,I,J)
      H1=XF(6,1,I,J)
      H1P=XF(6,2,I,J)
      H1PP=XF(6,3,I,J)
      H1PPP=XF(6,4,I,J)
      H1S=XF(6,5,I,J)
      H1SS=XF(6,6,I,J)
      H1SSS=XF(6,7,I,J)
      H1SP=XF(6,8,I,J)
      H1SSP=XF(6,9,I,J)
      H1SPP=XF(6,10,I,J)
      H2=XF(7,1,I,J)
      H2P=XF(7,2,I,J)
      H2PP=XF(7,3,I,J)
      H2PPP=XF(7,4,I,J)
      H2S=XF(7,5,I,J)
      H2SS=XF(7,6,I,J)
      H2SSS=XF(7,7,I,J)
      H2SP=XF(7,8,I,J)
      H2SSP=XF(7,9,I,J)
      H2SPP=XF(7,10,I,J)
      C=XF(8,1,I,J)
      CP=XF(8,2,I,J)
      CPP=XF(8,3,I,J)
      CPPP=XF(8,4,I,J)
      CS=XF(8,5,I,J)
      CSS=XF(8,6,I,J)
      CSSS=XF(8,7,I,J)
      CSP=XF(8,8,I,J)
      CSSP=XF(8,9,I,J)
      CSPP=XF(8,10,I,J)
677   FORMAT(' THE F',D15.7/3D15.7/3D15.7/3D15.7/' THE GO',
     C D15.7/3D15.7/3D15.7/3D15.7/' THE G1',D15.7/3D15.7/
     C 3D15.7/3D15.7/' THE G2',D15.7/3D15.7/3D15.7/3D15.7/
     C ' THE HO',D15.7/3D15.7/3D15.7/3D15.7/' THE H1',D15.7/
     C 3D15.7/3D15.7/3D15.7/' THE H2',D15.7/3D15.7/3D15.7/
     C 3D15.7/' THE C ',D15.7/3D15.7/3D15.7/3D15.7)
      IF(NI.EQ.2)WRITE(6,677) F,FP,FPP,FPPP,FS,FSS,FSSS,FSP,FSSP,FSPP,
     C GO,GOP,GOPP,GOPPP,GOS,GOSS,GOSSS,GOSP,GOSSP,GOSPP,
     C G1,G1P,G1PP,G1PPP,G1S,G1SS,G1SSS,G1SP,G1SSP,G1SPP,
     C G2,G2P,G2PP,G2PPP,G2S,G2SS,G2SSS,G2SP,G2SSP,G2SPP,
     C HO,HOP,HOPP,HOPPP,HOS,HOSS,HOSSS,HOSP,HOSSP,HOSPP,
     C H1,H1P,H1PP,H1PPP,H1S,H1SS,H1SSS,H1SP,H1SSP,H1SPP,
     C H2,H2P,H2PP,H2PPP,H2S,H2SS,H2SSS,H2SP,H2SSP,H2SPP,
     C C,CP,CPP,CPPP,CS,CSS,CSSS,CSP,CSSP,CSPP
      IF(IX.EQ.2)WRITE(4,677) F,FP,FPP,FPPP,FS,FSS,FSSS,FSP,FSSP,FSPP,
     C GO,GOP,GOPP,GOPPP,GOS,GOSS,GOSSS,GOSP,GOSSP,GOSPP,
```

```
      C G1,G1P,G1PP,G1PPP,G1S,G1SS,G1SSS,G1SP,G1SSP,G1SPP,
      C G2,G2P,G2PP,G2PPP,G2S,G2SS,G2SSS,G2SP,G2SSP,G2SPP,
      C H0,H0P,H0PP,H0PPP,H0S,H0SS,H0SSS,H0SP,H0SSP,H0SPP,
      C H1,H1P,H1PP,H1PPP,H1S,H1SS,H1SSS,H1SP,H1SSP,H1SPP,
      C H2,H2P,H2PP,H2PPP,H2S,H2SS,H2SSS,H2SP,H2SSP,H2SPP,
      C C,CP,CPP,CPPP,CS,CSS,CSSS,CSP,CSSP,CSPP
       IF(IX.EQ.2)WRITE(10,711) F,G0,G1,G2,H0,H1,H2,C
 711   FORMAT(4D20.9)
C RIGHT HAND SIDE OF THE EQUATION
       RGL=EP*( TO(1) +TO(7) +4*TO(8) +2*P2*TOP(8) +2*POK*U*TOP(9)
      C        +TU*TOS(9)/PK )
       RPP=EP*( 2*TOP(1) -U*TOS(1)/P2 +6*TO(2) +2*P2*TOP(2)
      C        +2*POK*U*TOP(3) +TU*TOS(3)/PK +2*TOP(7) -U*TOS(7)/P2 )
       RPLG=EP*( TO(3) +2*TOP(4) -U*TOS(4)/P2 +5*TO(5) +2*P2*TOP(5)
      C         +2*PKU*TOP(6) +TU*TOS(6)/POK +TOS(7)/POK )
       RE=EP*( TO(10) +4*TO(11)  +2*P2*TOP(11) +TOS(12)/POK +2*TOP(13)
      C       -U*TOS(13)/P2 +2*PKU*TOP(14) +TU*TOS(14)/POK )
       RPL=EP*( 5*TE(1) +2*P2*TEP(1) +2*POK*U*TEP(2) +TU*TES(2)/PK
      C        +2*TEP(5) -U*TES(5)/P2  )
       RSLN=EP*( TES(4)/POK +2*PKU*TEP(9) +TU*TES(9)/POK +TE(10)
      C         +4*TE(12) +2*P2*TEP(12) -TE(13) +TE(14) )
       RSALP=EP*(2*TEP(4) -U*TES(4)/P2 -5*TE(8) -2*P2*TEP(8) -TE(11)
      C          -2*POK*U*TEP(13) -TES(13)*TU/PK )
       RPLSN=EP*( 6*TE(6)  +2*P2*TEP(6) +2*PKU*TEP(7) +TU*TES(7)/POK
      C          +2*TEP(10) -U*TES(10)/P2 +TES(11)/POK +2*TEP(14)
      C          -U*TES(14)/P2 )
C LEFT HAND SIDE OF THE EQUATION
       LGL=-( 3*F/P2 +5*U*FS/P2 -TU*FSS/P2 -8*FP -4*P2*FPP
      C    -8*U*H0/P2  +4*TU*H0S/P2  +8*U*H0P
      C    -12*U*H1SP +4*TU*H1SS/P2  +20*H1P
      C    +4*TU*H1SSP +64*P2*H1PP  +16*P4*H1PPP -8*U*H1S/P2
      C    +(18*U2-6)*H2S/P2  -14*U*TU*H2SS/P2  +2*TU*TU*H2SSS/P2
      C    +(16-28*U2)*H2SP +4*U*TU*H2SSP +48*P2*U*H2PP
      C    +16*U*P4*H2PPP  +8*P2*TU*H2SPP )/PK
       LPP=( 9*F/P4  +23*U*FS/P4  -3*FSS/P4  +10*U2*FSS/P4
      C    -U*TU*FSSS/P4  -14*FP/P2  -18*U*FSP/P2  +2*TU*FSSP/P2
      C    +20*FPP -4*U*FSPP +8*P2*FPPP
      C    -64*U*H0/P4 +(30-86*U2)*H0S/P4  +26*U*TU*H0SS/P4
      C    -2*TU*TU*H0SSS/P4
      C    +64*U*H0P/P2  +(44*U2-16)*H0SP/P2  -4*U*TU*H0SSP/P2
      C    -32*U*H0PP -8*TU*H0SPP -16*P2*U*H0PPP
      C    +18*H1/P4  +46*U*H1S/P4  +(20*U2-6)*H1SS/P4
      C    -2*U*TU*H1SSS/P4  -28*H1P/P2  -36*U*H1SP/P2
      C    +4*TU*H1SSP/P2  +40*H1PP -8*U*H1SPP  +16*P2*H1PPP)/PK
       LPLG= -8*FS/P4  -7*U*FSS/P4  +8*FSP/R +4*FSPP +TU*FSSS/P4
      C      -8*H0/P4  -4*U*H0S/P4  +8*H0P/R -28*U*H0SP/R
      C      +4*TU*H0SSP/R  +64*H0PP +16*R*H0PPP
      C      +8*H1SP/R  -4*U*H1SS/P4  -8*H1S/P4
      C      +18*U*H2S/P4  +(10*U2-4*TU)*H2SS/P4  -2*U*TU*H2SSS/P4
      C      -28*U*H2SP/R  +48*H2PP  -8*U*H2SPP  +16*R*H2PPP
      C      +4*TU*H2SSP/R
       LE=-(8*H0/P4 +10*U*H0S/P4 -2*TU*H0SS/P4 -8*H0P/R -8*H0PP
      C    +8*H1S/P4  +10*U*H1SS/P4  -2*TU*H1SSS/P4  -8*H1SP/R
      C    -8*H1SPP
```

```
    C       +18*U*H2S/P4  +(10*U2-4*TU)*H2SS/P4  -2*U*TU*H2SSS/P4
    C       -28*U*H2SP/R  +48*H2PP  -8*U*H2SPP  +16*R*H2PPP
    C       +4*TU*H2SSP/R )
     LSLN=AKOP*(3*GOS/P2  +5*U*GOSS/P2 -TU*GOSSS/P2  -8*GOSP
    C      -4*P2*GOSPP
    C      -3*G2/P2  -5*U*G2S/P2  +TU*G2SS/P2  +8*G2P +4*P2*G2PP
    C      -3*C/P2  -5*U*CS/P2  +TU*CSS/P2  -2*CP +10*U*CSP
    C      -2*TU*CSSP  -28*P2*CPP -8*P4*CPPP )
     LSALP= 9*U*GOS/P4  -(2-7*U2)*GOSS/P4  -U*TU*GOSSS/P4
    C       -14*U*GOSP/R  +2*TU*GOSSP/R  +24*GOPP -4*U*GOSPP
    C       +8*R*GOPPP
    C       +4*G1/P4  +5*U*G1S/P4  -TU*G1SS/P4  -4*G1P/R  -4*G1PP
     LPLSN= AKOP*( 9*G1S/P4 +7*U*G1SS/P4 -TU*G1SSS/P4 -4*G1SP/P2
    C  -4*G1SPP
    C  +9*G2/P4 +23*U*G2S/P4 -(3.D0-10*U2)*G2SS/P4  -U*TU*G2SSS/P4
    C  -14*G2P/P2  -18*U*G2SP/P2  +2*TU*G2SSP/P2  +20*G2PP
    C  -4*U*G2SPP  +8*P2*G2PPP
    C  +9*C/P4  +23*U*CS/P4  -(3.D0-10*U2)*CSS/P4  -TU*U*CSSS/P4
    C  -14*CP/P2  -18*U*CSP/P2  +2*TU*CSSP/P2  +20*CPP
    C  -4*U*CSPP +8*P2*CPPP )
     LPL= 9*U*GOS/P4  -(2.D0-7*U2)*GOSS/P4  -U*TU*GOSSS/P4
    C     -14*U*GOSP/R  +2*TU*GOSSP/R  +24*GOPP  -4*U*GOSPP
    C     +8*R*GOPPP
    C     -12*G1/P4  -15*U*G1S/P4 +3*TU*G1SS/P4  +12*G1P/R
    C     -10*U*G1SP/R  +2*TU*G1SSP/R +36*G1PP  +8*R*G1PPP
    C     +9*U*G2/P4  -(8.D0-23*U2)*G2S/P4  -10*U*TU*G2SS/P4
    C     +TU*TU*G2SSS/P4  -14*U*G2P/R  +(8.D0-18*U2)*G2SP/R
    C     +2*U*TU*G2SSP/R  +20*U*G2PP  +4*TU*G2SPP  +8*R*U*G2PPP
C EROR IS THE SQUARE OF THE DIFFERENCES OF THE LEFT AND RIGHT SIDES OF
C THE EIGHT EQUATIONS.
      E(1)=2.0*DABS((LPL-RPL)/(LPL+RPL))
      E(2)=2.0*DABS((LSALP-RSALP)/(LSALP+RSALP))
      E(3)=2.0*DABS((LSLN-RSLN)/(LSLN+RSLN))
      E(4)=2.0*DABS((LPLSN-RPLSN)/(LPLSN+RPLSN))
      E(5)=2.0*DABS((LPP-RPP)/(LPP+RPP))
      E(6)=2.0*DABS((LGL-RGL)/(LGL+RGL))
      E(7)=2.0*DABS((LPLG-RPLG)/(LPLG+RPLG))
      E(8)=2.0*DABS((LE-RE)/(LE+RE))
      IOUT=1
      IF(IOUT.NE.1) GO TO 98
      IOUT=0
      IF(IOUT.EQ.1)WRITE(6,777) (E(IN),IN=1,8)
 777  FORMAT (' THE LEFT MINUS RIGHT',4D20.7)
C      WRITE (6,666) LPL,LSALP,LSLN,LPLSN
C      WRITE (6,667) RPL,RSALP,RSLN,RPLSN
C      WRITE (6,666) LPP,LGL,LPLG,LE
C      WRITE (6,667) RPP,RGL,RPLG,RE
 666  FORMAT(' LEFT ',4D14.5)
 667  FORMAT(' RIGHT',4D14.5)
      ALHSV1=LPP
      RHSV1=RPP
      ALHSV2=LPLG
      RHSV2=RPLG
      ALHSV3=LGL
```

```
      RHSV3=RGL
      ALHSV4=LE
      RHSV4=RE
      ALHSW1=LPLSN
      RHSW1=RPLSN
      ALHSW2=LSALP
      RHSW2=-RSALP
      ALHSW3=LSLN
      RHSW3=RSLN
      ALHSW4=LPL
      RHSW4=RPL
      RATV1=DABS((RHSV1-ALHSV1)/ALHSV1)*100.D0
      RATV2=DABS((RHSV2-ALHSV2)/ALHSV2)*100.D0
      RATV3=DABS((RHSV3-ALHSV3)/ALHSV3)*100.D0
      RATV4=DABS((RHSV4-ALHSV4)/ALHSV4)*100.D0
      RATW1=DABS((RHSW1-ALHSW1)/ALHSW1)*100.D0
      RATW2=DABS((RHSW2-ALHSW2)/ALHSW2)*100.D0
      RATW3=DABS((RHSW3-ALHSW3)/ALHSW3)*100.D0
      RATW4=DABS((ALHSW4-RHSW4)/ALHSW4)*100.D0
      WRITE(6,987)ALHSV1,RHSV1,RATV1
      WRITE(6,20)ALHSV2,RHSV2,RATV2
      WRITE(6,30)ALHSV3,RHSV3,RATV3
      WRITE(6,40)ALHSV4,RHSV4,RATV4
      WRITE(6,50)ALHSW1,RHSW1,RATW1
      WRITE(6,60)ALHSW2,RHSW2,RATW2
      WRITE(6,70)ALHSW3,RHSW3,RATW3
      WRITE(6,80)ALHSW4,RHSW4,RATW4
      IF(IX.EQ.2)WRITE(4,987)ALHSV1,RHSV1,RATV1
      IF(IX.EQ.2)WRITE(4,20)ALHSV2,RHSV2,RATV2
      IF(IX.EQ.2)WRITE(4,30)ALHSV3,RHSV3,RATV3
      IF(IX.EQ.2)WRITE(4,40)ALHSV4,RHSV4,RATV4
      IF(IX.EQ.2)WRITE(4,50)ALHSW1,RHSW1,RATW1
      IF(IX.EQ.2)WRITE(4,60)ALHSW2,RHSW2,RATW2
      IF(IX.EQ.2)WRITE(4,70)ALHSW3,RHSW3,RATW3
      IF(IX.EQ.2)WRITE(4,80)ALHSW4,RHSW4,RATW4
  987 FORMAT(1X,'EQ.1,LEFT=',E12.5,1X,'RIGHT=',E12.5,1X,
     C'ERROR=',F6.2,'%')
   20 FORMAT(1X,'EQ.2,LEFT=',E12.5,1X,'RIGHT=',E12.5,1X,
     C'ERROR=',F6.2,'%')
   30 FORMAT(1X,'EQ.3,LEFT=',E12.5,1X,'RIGHT=',E12.5,1X,
     C'ERROR=',F6.2,'%')
   40 FORMAT(1X,'EQ.4,LEFT=',E12.5,1X,'RIGHT=',E12.5,1X,
     C'ERROR=',F6.2,'%'/)
   50 FORMAT(1X,'EQ.5,LEFT=',E12.5,1X,'RIGHT=',E12.5,1X,
     C'ERROR=',F6.2,'%')
   60 FORMAT(1X,'EQ.6,LEFT=',E12.5,1X,'RIGHT=',E12.5,1X,
     C'ERROR=',F6.2,'%')
   70 FORMAT(1X,'EQ.7,LEFT=',E12.5,1X,'RIGHT=',E12.5,1X,
     C'ERROR=',F6.2,'%')
   80 FORMAT(1X,'EQ.8,LEFT=',E12.5,1X,'RIGHT=',E12.5,1X,
     C'ERROR=',F6.2,'%')
      IF(IX.EQ.2)WRITE (4,793) A,AP,APP,B,BP
  793 FORMAT (' A,AP,APP,B,BP',3D15.7/2D15.7)
   98 CONTINUE
```

```
      DIFF=0.0
      DO 99 IN=1,8
      DIFF=DIFF+E(IN)
   99 CONTINUE
      IF(IOUT.EQ.1) WRITE (6,400) I,J,DIFF
  400 FORMAT(' THE SUM OF THE DIFFERENCES AT ',2I3,D16.6)
      EROR(I,J)=DIFF
   10 CONTINUE
      FTBM=0.0
      DO 717 IN=1,LIM
      DO 717 JN=1,LOT
717   FTBM=FTBM+EROR(IN,JN)
      RETURN
      END
C THE FOLLOWING SUBROUTINES DEFINE THE TENSORS WHICH DEFINE THE
C RIGHT HAND SIDE OF THE EQUATIONS.
      SUBROUTINE TT(P2,K2,U,I,J)
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 MPD,MPDP,MPDU,K2
      COMMON/SIGNPK/SIGN
      COMMON/SUBTT/TE(14),TES(14),TEP(14),AOUT,APOUT,APP,BOUT,BPOUT,
     C            TO(14),TOS(14),TOP(14)
      COMMON/TWS/W1,W2,W3,W4,W5,W6,W7,W8,W9,W10,W11,W12,W13,W14
     C ,W1P,W2P,W3P,W4P,W5P,W6P,W7P,W8P,W9P,W10P,W11P,W12P,W13P,W14P
     C ,W1U,W2U,W3U,W4U,W5U,W6U,W7U,W8U,W9U,W10U,W11U,W12U,W13U,W14U
      COMMON/TVS/V1,V2,V3,V4,V5,V6,V7,V8,V9,V10,V11,V12,V13,V14
     C ,V1P,V2P,V3P,V4P,V5P,V6P,V7P,V8P,V9P,V10P,V11P,V12P,V13P,V14P
     C ,V1U,V2U,V3U,V4U,V5U,V6U,V7U,V8U,V9U,V10U,V11U,V12U,V13U,V14U
      COMMON/FUNS/XF(8,10,1,1)
C THIS SUBROUTINE GENERATES THE FUNCTIONS TE AND TO FOR THE R.H.S. OF THE EQUATI
      T1=1.D0
      T2=2.D0
      T3=3.D0
      T4=4.D0
      T5=5.D0
      T6=6.D0
      F=XF(1,1,I,J)
      FP=XF(1,2,I,J)
      FPP=XF(1,3,I,J)
      FS=XF(1,5,I,J)
      FSS=XF(1,6,I,J)
      FSP=XF(1,8,I,J)
      G0=XF(2,1,I,J)
      G0P=XF(2,2,I,J)
      G0PP=XF(2,3,I,J)
      G0S=XF(2,5,I,J)
      G0SS=XF(2,6,I,J)
      G0SP=XF(2,8,I,J)
      G1=XF(3,1,I,J)
      G1P=XF(3,2,I,J)
      G1PP=XF(3,3,I,J)
      G1S=XF(3,5,I,J)
      G1SS=XF(3,6,I,J)
      G1SP=XF(3,8,I,J)
```

```
      G2=XF(4,1,I,J)
      G2P=XF(4,2,I,J)
      G2PP=XF(4,3,I,J)
      G2S=XF(4,5,I,J)
      G2SS=XF(4,6,I,J)
      G2SP=XF(4,8,I,J)
      HO=XF(5,1,I,J)
      HOP=XF(5,2,I,J)
      HOPP=XF(5,3,I,J)
      HOS=XF(5,5,I,J)
      HOSS=XF(5,6,I,J)
      HOSP=XF(5,8,I,J)
      H1=XF(6,1,I,J)
      H1P=XF(6,2,I,J)
      H1PP=XF(6,3,I,J)
      H1S=XF(6,5,I,J)
      H1SS=XF(6,6,I,J)
      H1SP=XF(6,8,I,J)
      H2=XF(7,1,I,J)
      H2P=XF(7,2,I,J)
      H2PP=XF(7,3,I,J)
      H2S=XF(7,5,I,J)
      H2SS=XF(7,6,I,J)
      H2SP=XF(7,8,I,J)
      C=XF(8,1,I,J)
      CP=XF(8,2,I,J)
      CPP=XF(8,3,I,J)
      CS=XF(8,5,I,J)
      CSS=XF(8,6,I,J)
      CSP=XF(8,8,I,J)
      IKI=0
      IF(IKI.EQ.1)WRITE (6,100) F,FP,FS,G0,GOP,
     C GOS,G1,G1P,G1S,G2,
     C G2P,G2S,HO,HOP,HOS,H1,H1P,H1S,H2,H2P,H2S,C,CP,CS,P2,U
  100 FORMAT (' F,G,H,C,AS SEEN IN TT'/5D14.5/5D14.5/5D14.5/
     C 5D14.5/4D14.5/' P2,U IN TT',2D18.8)
      PK=SIGN*DSQRT(P2*K2)
      P4=P2*P2
      POK=DSQRT(P2/K2)
      AKOP=T1/POK
      P12=P2+K2+2*PK*U
      DU=T2*PK
      DP=T1+AKOP*U
      U2=U*U
      P4=P2*P2
      P6=P2*P4
      TU=1-U2
      A1=XA(P12)
      A2=XA(P2)
      DFF=100*DABS((A2-A1)/A2)
      WRITE(6,456) A1,A2,DFF,P12,P2
      WRITE(4,456) A1,A2,DFF,P12,P2
  456 FORMAT(' A1 AND A2',3D15.7,' %DIFF'/' P1 SQ AND P2 SQ',2D15.7)
      B1=XB(P12)
```

```
B2=XB(P2)
AP=XAP(P2)
BP=XBP(P2)
A1P=XAP(P12)*DP
A1U=XAP(P12)*DU
A2P=AP
AOUT=A2
BOUT=B2
APOUT=AP
BPOUT=BP
D1=A1*A1-P12*B1*B1
D2=A2*A2-P2*B2*B2
AD=A1/D1
BD=B1/D1
MPD=AP*B2/D2 -BP*A2/D2
PPD=AP*A2/D2 -P2*BP*B2/D2
BBD=B2*B2/D2
ABD=A2*B2/D2
APP=XAPP(P2)
D1P=T2*A1*A1P-DP*B1*B1
D1U=T2*A1*A1U -DU*B1*B1
D2P=T2*A2*A2P-B2*B2
ADP=A1P/D1 -A1*D1P/D1**2
ADU=A1U/D1 -A1*D1U/D1**2
AZ1=A1U/D1
AZ2=A1*D1U/D1**2
BDP=-B1*D1P/D1**2
BDU=-B1*D1U/D1**2
MPDP=APP*B2/D2 -AP*B2*D2P/D2**2
PPDP=APP*A2/D2 +AP*A2P/D2 -AP*A2*D2P/D2**2
BBDP=-B2*B2*D2P/D2**2
ABDP=A2P*B2/D2 -A2*B2*D2P/D2**2
ZE1=(-F/P2 +2*FP  -U*FS/P2 -2*PPD*F +F*BBD)/PK
C          +2*MPD*G0   +2*G1*MPD -G1*ABD/P2
C          +2*U*G2*MPD   +2*U*H0*BBD/PK -2*H1*BBD/PK
ZE1P=(-FP/P2 +2*FPP -U*FSP/P2 -2*PPD*FP +FP*BBD)/PK
C          +2*MPD*G0P   +2*G1P*MPD -G1P*ABD/P2
C          +2*U*G2P*MPD   +2*U*H0P*BBD/PK -2*H1P*BBD/PK +
C (1.5D0*F/P4 -FP/P2 +1.5D0*U*FS/P4 +PPD*F/P2 -0.5D0*F*BBD/P2)/PK
C +G1*ABD/P4 -U*H0*BBD/(PK*P2) +H1*BBD/(PK*P2)
C -2*PPDP*F/PK +F*BBDP/PK +2*MPDP*G0 +2*G1*MPDP -G1*ABDP/P2
C +2*U*G2*MPDP +2*U*H0*BBDP/PK -2*H1*BBDP/PK
ZE1U=(-FS/P2 +2*FSP -U*FSS/P2 -2*PPD*FS +FS*BBD)/PK
C    +2*MPD*G0S +2*G1S*MPD -G1S*ABD/P2
C    +2*U*G2S*MPD +2*U*H0S*BBD/PK -2*H1S*BBD/PK
C -FS/(P2*PK) +2*G2*MPD +2*H0*BBD/PK
ZE2=( FS/P2  -G2*ABD/POK  -2*H0*BBD  -2*H2*BBD  )
ZE2P=FSP/P2 -G2P*ABD/POK -2*H0P*BBD -2*H2P*BBD
C     -FS/P4 +0.5D0*G2*ABD/(P2*POK)
C     -G2*ABDP/POK -2*H0*BBDP -2*H2*BBDP
ZE2U=FSS/P2 -G2S*ABD/POK -2*H0S*BBD -2*H2S*BBD
ZE3=( C*ABD/POK +2*H2*BBD )
ZE3P=CP*ABD/POK +2*H2P*BBD -0.5D0*C*ABD/(POK*P2)
C    +C*ABDP/POK +2*H2*BBDP
```

```
 ZE3U=CS*ABD/POK +2*H2S*BBD
 ZE4=( -GO*ABD +2*H1/PK +2*POK*H1*BBD +2*POK*U*H2*BBD )
 ZE4P=-GOP*ABD +2*H1P/PK +2*POK*H1P*BBD +2*POK*U*H2P*BBD
C  -H1/(PK*P2) +H1*BBD/PK +U*H2*BBD/PK
C  -GO*ABDP +2*POK*H1*BBDP +2*POK*U*H2*BBDP
 ZE4U=-GOS*ABD +2*H1S/PK +2*POK*H1S*BBD +2*POK*U*H2S*BBD
C  +2*POK*H2*BBD
 ZE5=( F/PK  -GO*ABD  +2*H1*POK*BBD +2*POK*U*H2*BBD )
 ZE5P=FP/PK -GOP*ABD +2*H1P*POK*BBD +2*POK*U*H2P*BBD
C -0.5DO*F/(P2*PK) +H1*BBD/PK +U*H2*BBD/PK
C -GO*ABDP +2*H1*POK*BBDP +2*POK*U*H2*BBDP
 ZE5U=FS/PK -GOS*ABD +2*H1S*POK*BBD +2*POK*U*H2S*BBD
C    +2*POK*H2*BBD
 ZE6=(-2*G2*MPD/POK -4*HO/P4 +4*HOP/P2 -2*U*HOS/P4
C              -4*HO*PPD/P2 +2*C*MPD/POK )
 ZE6P=-2*G2P*MPD/POK -4*HOP/P4 +4*HOPP/P2 -2*U*HOSP/P4
C -4*HOP*PPD/P2 +2*CP*MPD/POK
C +G2*MPD/(POK*P2) +8*HO/P6 -4*HOP/P4 +4*U*HOS/P6
C +4*HO*PPD/P4 -C*MPD/(POK*P2)
C -2*G2*MPDP/POK -4*HO*PPDP/P2 +2*C*MPDP/POK
 ZE6U=-2*G2S*MPD/POK -4*HOS/P4 +4*HOSP/P2 -2*U*HOSS/P4
C -4*HOS*PPD/P2 +2*CS*MPD/POK -2*HOS/P4
 ZE7=(  2*HOS/(P2*PK)  )
 ZE7P=2*HOSP/(P2*PK) -3*HOS/(P4*PK)
 ZE7U=2*HOSS/(P2*PK)
 ZE8=(-2*GO*MPD  +2*H1/(P2*PK) -4*H1P/PK +2*U*H1S/(P2*PK)
C              +4*H1*PPD/PK -2*U*C*MPD  )
 ZE8P=-2*GOP*MPD +2*H1P/(P2*PK) -4*H1PP/PK +2*U*H1SP/(P2*PK)
C  +4*H1P*PPD/PK -2*U*CP*MPD
C  -3*H1/(P4*PK) +2*H1P/(P2*PK) -3*U*H1S/(P4*PK)
C  -2*H1*PPD/(P2*PK)
C  -2*GO*MPDP +4*H1*PPDP/PK -2*U*C*MPDP
 ZE8U=-2*GOS*MPD +2*H1S/(P2*PK) -4*H1SP/PK +2*U*H1SS/(P2*PK)
C  +4*H1S*PPD/PK -2*U*CS*MPD +2*H1S/(P2*PK) -2*C*MPD
 ZE9=( 2*H2S/PK )
 ZE9P=2*H2SP/PK -H2S/(P2*PK)
 ZE9U=2*H2SS/PK
 ZE10=( G2*ABD/POK +2*HO/P2 +2*HO*BBD  +2*H2*BBD )
 ZE10P=G2P*ABD/POK +2*HOP/P2 +2*HOP*BBD +2*H2P*BBD
C  -0.5DO*G2*ABD/(P2*POK) -2*HO/P4  +G2*ABDP/POK +2*HO*BBDP
C   +2*H2*BBDP
 ZE10U=G2S*ABD/POK +2*HOS/P2 +2*HOS*BBD +2*H2S*BBD
 ZE11=(  F*BBD/PK -G1*ABD/P2 +2*U*HO*BBD/PK -2*H1*BBD/PK)
 ZE11P=FP*BBD/PK -G1P*ABD/P2 +2*U*HOP*BBD/PK -2*H1P*BBD/PK
C  -0.5DO*F*BBD/(P2*PK) +G1*ABD/P4
C   -U*HO*BBD/(P2*PK) +H1*BBD/(P2*PK)
C  +F*BBDP/PK -G1*ABDP/P2 +2*U*HO*BBDP/PK -2*H1*BBDP/PK
 ZE11U=FS*BBD/PK -G1S*ABD/P2 +2*U*HOS*BBD/PK -2*H1S*BBD/PK
C  +2*HO*BBD/PK
 ZE12=( 4*H2P -2*U*H2S/P2 -4*H2*PPD +2*H2*BBD
C      -2*PK*C*MPD  +C*ABD/POK  )
 ZE12P=4*H2PP -2*U*H2SP/P2 -4*H2P*PPD +2*H2P*BBD
C  -2*PK*CP*MPD +CP*ABD/POK
C  +2*U*H2S/P4 -C*MPD/POK -0.5DO*C*ABD/(P2*POK)
```

```
C  -4*H2*PPDP +2*H2*BBDP -2*PK*C*MPDP +C*ABDP/POK
 ZE12U=4*H2SP -2*U*H2SS/P2 -4*H2S*PPD +2*H2S*BBD -2*PK*CS*MPD
C  +CS*ABD/POK -2*H2S/P2
 ZE13=( -2*H1S/P2 +2*H2*BBD +C*ABD/POK )
 ZE13P=-2*H1SP/P2 +2*H2P*BBD +CP*ABD/POK +2*H1S/P4 -0.5D0*C*ABD
C  /(P2*POK) +2*H2*BBDP +C*ABDP/POK
 ZE13U=-2*H1SS/P2 +2*H2S*BBD +CS*ABD/POK
 ZE14=( 2*H0/P2 -2*H2*BBD -C*ABD/POK )
 ZE14P=2*H0P/P2 -2*H2P*BBD -CP*ABD/POK -2*H0/P4
C +0.5D0*C*ABD/(P2*POK)
C  -2*H2*BBDP -C*ABDP/POK
 ZE14U=2*H0S/P2 -2*H2S*BBD -CS*ABD/POK
 Z1= (-G0*BBD +G1/P2 +2*H1*ABD/PK -U*C*BBD )
 Z1P=-G0P*BBD +G1P/P2 +2*H1P*ABD/PK -U*CP*BBD
C  -G1/P4 -H1*ABD/(P2*PK) -G0*BBDP +2*H1*ABDP/PK -U*C*BBDP
 Z1U=-G0S*BBD +G1S/P2 +2*H1S*ABD/PK -U*CS*BBD -C*BBD
 Z2= ( 2*F*MPD/PK -2*G1/P4 +2*G1P/P2 -U*G1S/P4
C          -2*G1*PPD/P2  +4*U*H0*MPD/PK -4*H1*MPD/PK )
 Z2P=2*FP*MPD/PK -2*G1P/P4 +2*G1PP/P2 -U*G1SP/P4
C -2*G1P*PPD/P2 +4*U*H0P*MPD/PK -4*H1P*MPD/PK
C  -F*MPD/(P2*PK) +4*G1/P6 -2*G1P/P4 +2*U*G1S/P6
C  +2*G1*PPD/P4 -2*U*H0*MPD/(P2*PK) +2*H1*MPD/(P2*PK)
C  +2*F*MPDP/PK -2*G1*PPDP/P2 +4*U*H0*MPDP/PK -4*H1*MPDP/PK
 Z2U=2*FS*MPD/PK -2*G1S/P4 +2*G1SP/P2 -U*G1SS/P4
C  -2*G1S*PPD/P2 +4*U*H0S*MPD/PK -4*H1S*MPD/PK
C  -G1S/P4 +4*H0*MPD/PK
 Z3= ( G1S/(P2*POK) -G2*BBD/POK -2*H0*ABD/P2 +C*BBD/POK )
 Z3P=G1SP/(P2*POK) -G2P*BBD/POK -2*H0P*ABD/P2 +CP*BBD/POK
C-1.5D0*G1S/(P4*POK) +0.5D0*G2*BBD/(P2*POK)
C   +2*H0*ABD/P4 -0.5D0*C*BBD/(P2*POK)
C   -G2*BBDP/POK -2*H0*ABDP/P2 +C*BBDP/POK
 Z3U=G1SS/(P2*POK) -G2S*BBD/POK -2*H0S*ABD/P2 +CS*BBD/POK
 Z4=( G2/POK +2*H2*ABD +PK*C*BBD )
 Z4P=G2P/POK +2*H2P*ABD +PK*CP*BBD -0.5D0*G2/(P2*POK)
C   +0.5D0*C*BBD/POK
C  +2*H2*ABDP +PK*C*BBDP
 Z4U=G2S/POK +2*H2S*ABD +PK*CS*BBD
 Z5= (-G2/(P2*POK) -U*G2S/(P2*POK) +G2*BBD/POK +2*G2P/POK
C           -2*G2*PPD/POK +2*H0*ABD/P2 -4*H0*MPD -4*H2*MPD
C           -C*BBD/POK )
 Z5P=-G2P/(P2*POK) -U*G2SP/(P2*POK) +G2P*BBD/POK
C   +2*G2PP/POK -2*G2P*PPD/POK
C  +2*H0P*ABD/P2 -4*H0P*MPD -4*H2P*MPD -CP*BBD/POK
C  +1.5D0*G2/(P4*POK) +1.5D0*U*G2S/(P4*POK)
C   -0.5D0*G2*BBD/(P2*POK) -G2P/(P2*POK)
C  +G2*PPD/(P2*POK) -2*H0*ABD/P4 +0.5D0*C*BBD/(P2*POK)
C  +G2*BBDP/POK -2*G2*PPDP/POK +2*H0*ABDP/P2 -4*H0*MPDP
C  -4*H2*MPDP -C*BBDP/POK
 Z5U=-G2S/(P2*POK) -U*G2SS/(P2*POK) +G2S*BBD/POK +2*G2SP/POK
C  -2*G2S*PPD/POK +2*H0S*ABD/P2 -4*H0S*MPD -4*H2S*MPD
C   -CS*BBD/POK
C  -G2S/(P2*POK)
 Z6= (G2S/P2)
 Z6P=G2SP/P2 -G2S/P4
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```
 Z6U=G2SS/P2
 Z7= (-F*ABD/PK +G0*BBD +G1/P2 +G1*BBD +U*G2*BBD)
 Z7P=-FP*ABD/PK +G0P*BBD +G1P/P2 +G1P*BBD +U*G2P*BBD
C +0.5D0*F*ABD/(P2*PK) -G1/P4  -F*ABDP/PK +G0*BBDP
C +G1*BBDP +U*G2*BBDP
 Z7U=-FS*ABD/PK +G0S*BBD +G1S/P2 +G1S*BBD +U*G2S*BBD
C   +G2*BBD
 Z8=(2*G0P -U*G0S/P2 -2*G0*PPD +G0*BBD +4*POK*H1*MPD
C            -2*H1*ABD/PK +4*H2*POK*U*MPD +U*C*BBD )
 Z8P=2*G0PP -U*G0SP/P2 -2*G0P*PPD +G0P*BBD +4*POK*H1P*MPD
C  -2*H1P*ABD/PK +4*H2P*POK*U*MPD +U*CP*BBD
C +U*G0S/P4 +2*H1*MPD/PK +H1*ABD/(P2*PK) +2*H2*U*MPD/PK
C -2*G0*PPDP +G0*BBDP +4*POK*H1*MPDP -2*H1*ABDP/PK
C +4*H2*POK*U*MPDP +U*C*BBDP
 Z8U=2*G0SP -U*G0SS/P2 -2*G0S*PPD +G0S*BBD +4*POK*H1S*MPD
C -2*H1S*ABD/PK +4*H2S*POK*U*MPD +U*CS*BBD
C -G0S/P2 +4*POK*H2*MPD +C*BBD
 Z9= (G0S/POK -2*H2*ABD -PK*C*BBD )
 Z9P=G0SP/POK -2*H2P*ABD -PK*CP*BBD
C    -0.5D0*G0S/(P2*POK) -0.5D0*C*BBD/POK
C  -2*H2*ABDP -PK*C*BBDP
 Z9U=G0SS/POK -2*H2S*ABD -PK*CS*BBD
 Z10=(-G2*BBD/POK -2*H0*ABD/P2  +C*BBD/POK )
 Z10P=-G2P*BBD/POK -2*H0P*ABD/P2 +CP*BBD/POK
C  +0.5D0*G2*BBD/(P2*POK) +2*H0*ABD/P4 -0.5D0*C*BBD/(P2*POK)
C  -G2*BBDP/POK -2*H0*ABDP/P2 +C*BBDP/POK
 Z10U=-G2S*BBD/POK -2*H0S*ABD/P2 +CS*BBD/POK
 Z11=( -4*H2*MPD -C/(P2*POK) +2*CP/POK -U*CS/(P2*POK)
C      -2*C*PPD/POK   )
 Z11P=-4*H2P*MPD -CP/(P2*POK) +2*CPP/POK -U*CSP/(P2*POK)
C  -2*CP*PPD/POK +1.5D0*C/(P4*POK) -CP/(P2*POK)
C   +1.5D0*U*CS/(P4*POK)
C  +C*PPD/(P2*POK) -4*H2*MPDP -2*C*PPDP/POK
 Z11U=-4*H2S*MPD -CS/(P2*POK) +2*CSP/POK -U*CSS/(P2*POK)
C   -2*CS*PPD/POK
C  -CS/(P2*POK)
 Z12=( G0*BBD -2*H1*ABD/PK +U*C*BBD )
 Z12P=G0P*BBD -2*H1P*ABD/PK +U*CP*BBD +H1*ABD/(P2*PK)
C  +G0*BBDP -2*H1*ABDP/PK +U*C*BBDP
 Z12U=G0S*BBD -2*H1S*ABD/PK +U*CS*BBD +C*BBD
 Z14=( CS/P2 )
 Z14P=CSP/P2 -CS/P4
 Z14U=CSS/P2
 Z13=(2*H2*ABD +C/POK +PK*C*BBD )
 Z13P=2*H2P*ABD +CP/POK +PK*CP*BBD -0.5D0*C/(P2*POK)
C   +0.5D0*C*BBD/POK
C  +2*H2*ABDP +PK*C*BBDP
 Z13U=2*H2S*ABD +CS/POK +PK*CS*BBD
 V1=AD*Z1    +BD*(-ZE5 +POK*U*ZE14 +ZE14)
 V1P=AD*Z1P +ADP*Z1 +BDP*(-ZE5 +POK*U*ZE14 +ZE14)
C +BD*(-ZE5P +POK*U*ZE14P +ZE14P +0.5D0*U*ZE14/PK)
 V1U=AD*Z1U+ADU*Z1
C  +BD*(-ZE5U +POK*ZE14 +POK*U*ZE14U +ZE14U)
C  +BDU*(-ZE5 +POK*U*ZE14 +ZE14)
```

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 V2=AD*Z2  +BD*(-ZE1 +POK*U*ZE6 +ZE8 +ZE11 +ZE6 )
 V2P=ADP*Z2 +Z2P*AD +BD*(-ZE1P +0.5D0*U*ZE6/PK +POK*U*ZE6P
C +ZE8P+ZE11P +ZE6P ) +BDP*(-ZE1+POK*U*ZE6+ZE8+ZE11 +ZE6)
 V2U=ADU*Z2 +AD*Z2U +BDU*(-ZE1+POK*U*ZE6+ZE8+ZE11+ZE6)
C +BD*(-ZE1U+POK*ZE6+POK*U*ZE6U+ZE8U+ZE11U+ZE6U)
 V3=AD*Z3  +BD*(-ZE2 +PK*U*ZE7 +ZE13 +K2*ZE7 +K2*ZE11 )
 V3P=AD*Z3P+ADP*Z3 +BDP*(-ZE2 +PK*U*ZE7 +ZE13 +K2*ZE7 +K2*ZE11 )
C +BD*(-ZE2P+PK*U*ZE7P +0.5D0*U*ZE7/POK +ZE13P
C   +K2*ZE7P+K2*ZE11P)
 V3U=ADU*Z3+AD*Z3U +BDU*(-ZE2 +PK*U*ZE7 +ZE13 +K2*ZE7 +K2*ZE11 )
C  +BD*(-ZE2U+PK*ZE7+PK*U*ZE7U+ZE13U +K2*ZE7U +K2*ZE11U)
 V4=AD*Z4  +BD*(-P2*ZE14 -K2*ZE5 -PK*U*ZE14 )
 V4P=ADP*Z4+Z4P*AD +BDP*(-P2*ZE14 -K2*ZE5 -PK*U*ZE14 )
C +BD*(-P2*ZE14P -ZE14 -K2*ZE5P -0.5D0*U*ZE14/POK -PK*U*ZE14P)
 V4U=ADU*Z4+Z4U*AD +BDU*(-P2*ZE14 -K2*ZE5 -PK*U*ZE14)
C  +BD*(-P2*ZE14U-K2*ZE5U -PK*ZE14 -PK*U*ZE14U)
 V5=AD*Z5 +BD*(-P2*ZE6 -ZE10 -ZE12 -K2*ZE1 -PK*U*ZE6 )
 V5P=ADP*Z5+Z5P*AD+BDP*(-P2*ZE6 -ZE10 -ZE12 -K2*ZE1 -PK*U*ZE6)
C +BD*(-ZE6-P2*ZE6P-ZE10P-ZE12P-K2*ZE1P-PK*U*ZE6P
C   -U*0.5D0*ZE6/POK )
 V5U=ADU*Z5+Z5U*AD+BDU*(-P2*ZE6 -ZE10 -ZE12 -K2*ZE1 -PK*U*ZE6)
C  +BD*(-P2*ZE6U-ZE10U-ZE12U-K2*ZE1U-PK*ZE6 -PK*U*ZE6U)
 V6=AD*Z6 +BD*(-(P2+PK*U)*ZE7 -ZE9 -ZE2 -ZE10 )
 V6P=ADP*Z6+Z6P*AD+BDP*(-(P2+PK*U)*ZE7 -ZE9 -ZE2 -ZE10 )
C +BD*( -ZE7 -U*0.5D0*ZE7/POK -(P2+PK*U)*ZE7P -ZE9P -ZE2P
C     -ZE10P )
 V6U=AD*Z6U+ADU*Z6+BDU*(-(P2+PK*U)*ZE7 -ZE9 -ZE2 -ZE10 )
C +BD*(-PK*ZE7 -(P2+PK*U)*ZE7U -ZE9U -ZE2U-ZE10U)
 V7=AD*Z7 +BD*(-ZE4 +POK*U*ZE10 -(P2+PK*U)*ZE11 +ZE10 )
 V7U=ADU*Z7+AD*Z7U +BDU*(-ZE4 +POK*U*ZE10
C   -(P2+PK*U)*ZE11 +ZE10)
C  +BD*(-ZE4U +POK*ZE10 +POK*U*ZE10U
C   -PK*ZE11 -(P2+PK*U)*ZE11U
C  +ZE10U)
 V7P=AD*Z7P+ADP*Z7 +BDP*(-ZE4 +POK*U*ZE10 -(P2+PK*U)*ZE11
C   +ZE10 )
C  +BD*(-ZE4P+0.5D0*U*ZE10/PK +POK*U*ZE10P -(P2+PK*U)*ZE11P
C  -ZE11 -0.5D0*U*ZE11/POK +ZE10P )
 V8=AD*Z8 +BD*( ZE4-(P2+PK*U)*ZE8 +(1.D0 +POK*U)*ZE12 )
 V8P=ADP*Z8+AD*Z8P +BDP*( ZE4-(P2+PK*U)*ZE8
C  +(1.D0+POK*U)*ZE12 )
C  +BD*( ZE4P -ZE8 -0.5D0*U*ZE8/POK -(P2+PK*U)*ZE8P +
C        (1.D0+POK*U)*ZE12P +U*0.5D0*ZE12/PK )
 V8U=ADU*Z8+AD*Z8U +BDU*( ZE4 -(P2+PK*U)*ZE8
C   +(1.D0+POK*U)*ZE12)
C  +BD*( ZE4U -PK*ZE8 -(P2+PK*U)*ZE8U +(1.D0+POK*U)*ZE12U
C        +POK*ZE12)
 V9=AD*Z9 +BD*((K2+PK*U)*ZE9 -(P2+PK*U)*ZE13 +K2*ZE4 )
 V9P=ADP*Z9 +AD*Z9P +BDP*((K2+PK*U)*ZE9
C   -(P2+PK*U)*ZE13 +K2*ZE4)
C  +BD*(0.5D0*U*ZE9/POK +(K2+PK*U)*ZE9P -ZE13 -(P2+PK*U)*ZE13P
C  +K2*ZE4P -U*0.5D0*ZE13/POK )
 V9U=ADU*Z9+Z9U*AD +BDU*((K2+PK*U)*ZE9 -(P2+PK*U)*ZE13 +K2*ZE4)
C  +BD*( (K2+PK*U)*ZE9U +PK*ZE9 -(P2+PK*U)*ZE13U
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C  -PK*ZE13 +K2*ZE4U)
 V10=AD*Z10 +BD*( ZE3 +ZE10 +K2*ZE11 )
 V10P=AD*Z10P+ADP*Z10 +BDP*( ZE3 +ZE10 +K2*ZE11)
C  +BD*( ZE3P +ZE10P +K2*ZE11P )
 V10U=AD*Z10U +ADU*Z10 +BDU*( ZE3 +ZE10 +K2*ZE11 )
C  +BD*( ZE3U +ZE10U +K2*ZE11U )
 V11=AD*Z11 +BD*(ZE12 -ZE3 +K2*ZE8)
 V11P=ADP*Z11 +AD*Z11P +BDP*(ZE12 -ZE3 +K2*ZE8)
C  +BD*(ZE12P -ZE3P +K2*ZE8P)
 V11U=ADU*Z11 +AD*Z11U +BDU*(ZE12 -ZE3 +K2*ZE8)
C +BD*(ZE12U -ZE3U +K2*ZE8U)
 V12=AD*Z12   +BD*(+(1.D0+POK*U)*ZE3 +ZE4 )
 V12P=ADP*Z12 +AD*Z12P +BDP*( (1.D0+POK*U)*ZE3 +ZE4)
C  +BD*((1.D0+POK*U)*ZE3P +0.5D0*U*ZE3/PK +ZE4P)
 V12U=ADU*Z12 +AD*Z12U +BDU*((1.D0+POK*U)*ZE3 +ZE4)
C  +BD*((1.D0+POK*U)*ZE3U +POK*ZE3 +ZE4U)
 V13=AD*Z13 +BD*(-K2*ZE4+(P2+PK*U)*ZE3)
 V13P=ADP*Z13 +Z13P*AD +BD*(-K2*ZE4P +ZE3 +0.5D0*U*ZE3/POK
C  +(P2+PK*U)*ZE3P) +BDP*(-K2*ZE4+(P2+PK*U)*ZE3)
 V13U=ADU*Z13 +AD*Z13U +BDU*( -K2*ZE4 +(P2+PK*U)*ZE3)
C  +BD*(-K2*ZE4U +(P2+PK*U)*ZE3U +PK*ZE3)
 V14=AD*Z14 +BD*(ZE9 +ZE13  -ZE3)
 V14P=ADP*Z14+AD*Z14P +BDP*(ZE9 +ZE13 -ZE3)
C  +BD*(ZE9P +ZE13P -ZE3P)
 V14U=ADU*Z14 +AD*Z14U +BDU*(ZE9+ZE13 -ZE3)
C    +BD*(ZE9U +ZE13U -ZE3U)
 W1=AD*ZE1  +BD*(-(P2+PK*U)*Z2 -(1.D0+POK*U)*Z5 -Z7 -Z8 )
 W1P=AD*ZE1P+ADP*ZE1 +BD*(-Z2 -U*0.5D0*Z2/POK -(P2+PK*U)*Z2P
C  -0.5D0*U*Z5/PK -(1.D0+POK*U)*Z5P -Z7P -Z8P)
C    +BDP*(-(P2+PK*U)*Z2
C  -(1.D0+POK*U)*Z5 -Z7 -Z8)
 W1U=ADU*ZE1 +AD*ZE1U +BDU*(-(P2+PK*U)*Z2 -(1.D0+POK*U)*Z5 -Z7 -Z8)
C  +BD*(-PK*Z2 -(P2+PK*U)*Z2U -POK*Z5 -(1.D0+POK*U)*Z5U -Z7U -Z8U)
 W2=AD*ZE2  +BD*(-(P2+PK*U)*Z3 -(K2+PK*U)*Z6 -K2*Z7 -Z9 )
 W2P=ADP*ZE2 +AD*ZE2P
C +BDP*(-(P2+PK*U)*Z3 -(K2+PK*U)*Z6 -K2*Z7 -Z9)
C  +BD*(-Z3 -0.5D0*U*Z3/POK -(P2+PK*U)*Z3P -0.5D0*U*Z6/POK
C       -(K2+PK*U)*Z6P -K2*Z7P -Z9P )
 W2U=ADU*ZE2 +AD*ZE2U +BDU*(-(P2+PK*U)*Z3 -(K2+PK*U)*Z6
C  -K2*Z7-Z9)
C    +BD*(-PK*Z3 -(P2+PK*U)*Z3U -(K2+PK*U)*Z6U -PK*Z6 -K2*Z7U -Z9U)
 W3=AD*ZE3 +BD*( Z13 +K2*Z12)
 W3P=ADP*ZE3 +AD*ZE3P +BDP*(Z13 +K2*Z12 ) +BD*(Z13P+K2*Z12P)


 W3U=ADU*ZE3 +AD*ZE3U +BDU*(Z13+K2*Z12 ) +BD*(Z13U+K2*Z12U)
 W4=AD*ZE4+BD*(+(P2+PK*U)*Z12  -(1.D0+POK*U)*Z13 )
 W4P=ADP*ZE4 +AD*ZE4P +BDP*((P2+PK*U)*Z12 -(1.D0+POK*U)*Z13)
C    +BD*( Z12 +0.5D0*U*Z12/POK +(P2+PK*U)*Z12P -(1.D0+POK*U)*Z13P
C         -U*0.5D0*Z13/PK )
 W4U=ADU*ZE4 +AD*ZE4U +BDU*((P2+PK*U)*Z12 -(1.D0+POK*U)*Z13)
C  +BD*(PK*Z12 +(P2+PK*U)*Z12U -(1.D0+POK*U)*Z13U -POK*Z13 )
 W5=AD*ZE5 +BD*(-(P2+PK*U)*Z1 -(1.D0+POK*U)*Z4 )
 W5P=ADP*ZE5 +AD*ZE5P +BDP*(-(P2+PK*U)*Z1 -(1.D0+POK*U)*Z4)
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C  +BD*(-Z1 -0.5D0*U*Z1/POK -(P2+PK*U)*Z1P -(1.D0+POK*U)*Z4P
C        -0.5D0*U*Z4/PK )
 W5U=ADU*ZE5+AD*ZE5U +BDU*(-(P2+PK*U)*Z1 -(1.D0+POK*U)*Z4)
C  +BD*(-PK*Z1 -(P2+PK*U)*Z1U -(1.D0+POK*U)*Z4U -POK*Z4)
 W6=AD*ZE6+BD*(K2*Z2 -Z5 -Z10 -Z11 )
 W6P=ADP*ZE6 +AD*ZE6P +BDP*( K2*Z2 -Z5-Z10-Z11)
C   +BD*(K2*Z2P -Z5P -Z10P -Z11P)
 W6U=ADU*ZE6 +AD*ZE6U +BDU*( K2*Z2 -Z5 -Z10 -Z11)
C   +BD*( K2*Z2U -Z5U -Z10U -Z11U)
 W7=AD*ZE7+BD*( Z3 -Z6 -Z10 -Z14 )
 W7P=ADP*ZE7 +AD*ZE7P +BDP*( Z3 -Z6 -Z10 -Z14)
C   +BD*( Z3P -Z6P -Z10P -Z14P )
 W7U=ADU*ZE7 +AD*ZE7U +BDU*( Z3 -Z6 -Z10 -Z14 )
C   +BD*( Z3U -Z6U -Z10U -Z14U)
 W8=AD*ZE8 +BD*(-Z8 +(1.D0+POK*U)*Z11 +Z12  )
 W8P=ADP*ZE8 +AD*ZE8P +BDP*(-Z8 +(1.D0+POK*U)*Z11 +Z12 )
C +BD*( -Z8P +(1.D0+POK*U)*Z11P +0.5D0*U*Z11/PK +Z12P)
 W8U=ADU*ZE8 +AD*ZE8U +BDU*( -Z8 +(1.D0+POK*U)*Z11 +Z12)
C  +BD*( -Z8U +POK*Z11 +(1.D0+POK*U)*Z11U +Z12U )
 W9=AD*ZE9 +BD*( Z9 +(P2+PK*U)*Z14 +Z13 )
 W9P=ADP*ZE9 +AD*ZE9P +BDP*( Z9 +(P2+PK*U)*Z14 +Z13)
C   +BD*( Z9P +Z14 +U*0.5D0*Z14/POK +(P2+PK*U)*Z14P +Z13P)
 W9U=ADU*ZE9 +AD*ZE9U +BDU*( Z9 +(P2+PK*U)*Z14 +Z13)
C   +BD*(  Z9U +(P2+PK*U)*Z14U +PK*Z14 +Z13U)
 W10=AD*ZE10 +BD*(K2*Z7 +(P2+PK*U)*Z10 -Z13 )
 W10P=ADP*ZE10 +AD*ZE10P +BDP*(K2*Z7 +(P2+PK*U)*Z10-Z13)
C +BD*(K2*Z7P +Z10 +0.5D0*U*Z10/POK +(P2+PK*U)*Z10P -Z13P)
 W10U=ADU*ZE10 +AD*ZE10U +BDU*(K2*Z7 +(P2+PK*U)*Z10 -Z13)
C  +BD*(K2*Z7U +PK*Z10 +(P2+PK*U)*Z10U -Z13U)
 W11=AD*ZE11 +BD*(-Z7 +(1.D0+POK*U)*Z10  -Z12 )
 W11P=ADP*ZE11 +AD*ZE11P +BDP*(-Z7 +(1.D0+POK*U)*Z10 -Z12)
C  +BD*(-Z7P +(1.D0+POK*U)*Z10P +0.5D0*U*Z10/PK -Z12P)
 W11U=ADU*ZE11 +AD*ZE11U +BDU*( -Z7 +(1.D0+POK*U)*Z10 -Z12)
C  +BD*( -Z7U +(1.D0+POK*U)*Z10U +POK*Z10 -Z12U )
 W12=AD*ZE12    +BD*(K2*Z8 +(P2+PK*U)*Z11 +Z13 )
 W12P=ADP*ZE12 +AD*ZE12P +BDP*(K2*Z8 +(P2+PK*U)*Z11 +Z13)
C   +BD*( K2*Z8P +Z11 +0.5D0*U*Z11/POK +(P2+PK*U)*Z11P +Z13P)
 W12U=ADU*ZE12 +AD*ZE12U +BDU*( K2*Z8 +(P2+PK*U)*Z11 +Z13)
C   +BD*( K2*Z8U +PK*Z11 +(P2+PK*U)*Z11U +Z13U )
 W13=AD*ZE13 +BD*(-Z9 +(K2+PK*U)*Z14 +K2*Z12 )
 W13P=AD*ZE13P+ADP*ZE13 +BDP*(-Z9 +(K2+PK*U)*Z14 +K2*Z12)
C   +BD*(-Z9P +0.5D0*U*Z14/POK +(K2+PK*U)*Z14P +K2*Z12P )
 W13U=ADU*ZE13 +AD*ZE13U +BDU*( -Z9 +(K2+PK*U)*Z14 +K2*Z12)
C   +BD*(-Z9U +PK*Z14 +(K2+PK*U)*Z14U +K2*Z12U)
 W14=AD*ZE14 +BD*( K2*Z1 -Z4 )
 W14P=ADP*ZE14 +AD*ZE14P +BDP*(K2*Z1-Z4) +BD*(K2*Z1P-Z4P)
 W14U=ADU*ZE14 +AD*ZE14U +BDU*( K2*Z1-Z4) +BD*(K2*Z1U-Z4U)
 CALL TETO(P2,U,K2)
 RETURN
 END
 SUBROUTINE TETO(P2,U,K2)
 IMPLICIT REAL*8 (A-H,O-Z)
 REAL*8 K2
 COMMON/SIGNPK/SIGN
```

```
 COMMON/SUBTT/TE(14),TES(14),TEP(14),AOUT,APOUT,APP,BOUT,BPOUT,
C             TO(14),TOS(14),TOP(14)
 COMMON/TWS/W1,W2,W3,W4,W5,W6,W7,W8,W9,W10,W11,W12,W13,W14
C ,W1P,W2P,W3P,W4P,W5P,W6P,W7P,W8P,W9P,W10P,W11P,W12P,W13P,W14P
C ,W1U,W2U,W3U,W4U,W5U,W6U,W7U,W8U,W9U,W10U,W11U,W12U,W13U,W14U
 COMMON/TVS/V1,V2,V3,V4,V5,V6,V7,V8,V9,V10,V11,V12,V13,V14
C ,V1P,V2P,V3P,V4P,V5P,V6P,V7P,V8P,V9P,V10P,V11P,V12P,V13P,V14P
C ,V1U,V2U,V3U,V4U,V5U,V6U,V7U,V8U,V9U,V10U,V11U,V12U,V13U,V14U
 P4=P2*P2
 P6=P2*P4
 PK=SIGN*DSQRT(P2*K2)
 POK=DSQRT(P2/K2)
 TE(1)=( 1.5D0*W1 -0.5D0*POK*U*W6 -0.5D0*W7 -0.5D0*W8 -1.5D0*W11)
 TEP(1)=(1.5D0*W1P -0.5D0*POK*U*W6P -0.25D0*U*W6/PK -0.5D0*W7P
C       -0.5D0*W8P -1.5D0*W11P )
 TES(1)=1.5D0*W1U -0.5D0*POK*W6 -0.5D0*POK*U*W6U -0.5D0*W7U
C       -0.5D0*W8U -1.5D0*W11U
 TE(2)=( 1.5D0*W2 +0.5D0*P2*W6 +0.5D0*PK*U*W7 +1.5D0*W10
C         +0.5D0*W12 +0.5D0*W14 )
 TEP(2)=0.5D0*( 3*W2P +W6 +P2*W6P +PK*U*W7P +0.5D0*U*W7/POK
C       +3*W10P +W12P +W14P )
 TES(2)=0.5D0*( 3*W2U +P2*W6U +PK*U*W7U +PK*W7
C  +3*W10U +W12U +W14U)
 TE(3)=( 1.5D0*W3 -0.5D0*W12 -0.5D0*W13 +0.5D0*W14 )
 TEP(3)=0.5D0*( 3*W3P -W12P -W13P +W14P)
 TES(3)=0.5D0*( 3*W3U -W12U -W13U +W14U)
 TE(4)=( 2.5D0*W4 -0.5D0*W5 +0.5D0*W9 +0.5D0*POK*U*W12
C         -0.5D0*P2*W8 -0.5D0*POK*U*W13 )
 TEP(4)=0.5D0*(5*W4P -W5P +W9P +POK*U*W12P +0.5D0*U*W12/PK
C             -P2*W8P -W8 -POK*U*W13P -0.5D0*U*W13/PK )
 TES(4)=0.5D0*(5*W4U -W5U +W9U +POK*U*W12U +POK*W12 -P2*W8U
C          -POK*U*W13U -POK*W13 )
 TE(5)=( 1.5D0*W5 -1.5D0*W4 -0.5D0*POK*U*W12 +0.5D0*P2*W8
C         +0.5D0*POK*U*W13 -0.5D0*W9 )
 TEP(5)=0.5D0*(3*W5P -3*W4P -POK*U*W12P -U*0.5D0*W12/PK +W8
C             +P2*W8P +POK*U*W13P +0.5D0*U*W13/PK -W9P )
 TES(5)= 0.5D0*( 3*W5U -3*W4U -POK*W12 -POK*U*W12U +P2*W8U
C          +POK*W13 +POK*U*W13U -W9U )
 TE(6)=W6
 TEP(6)=W6P
 TES(6)=W6U
 TE(7)=W7
 TEP(7)=W7P
 TES(7)=W7U
 TE(8)=W8
 TEP(8)=W8P
 TES(8)=W8U
 TE(9)=W9
 TEP(9)=W9P
 TES(9)=W9U
 TE(10)=( 2.5D0*W10 +0.5D0*W2 +0.5D0*P2*W6 +0.5D0*PK*U*W7
C          +0.5D0*W14 +0.5D0*W12 )
 TEP(10)=0.5D0*( 5*W10P +W2P +W6 +P2*W6P +PK*U*W7P
C    +0.5D0*U*W7/POK +W14P +W12P )
```

```
 TES(10)=0.5D0*( 5*W10U +W2U +P2*W6U +PK*U*W7U +PK*W7
C     +W14U +W12U )
 TE(11)=(2.5D0*W11 -0.5D0*W1 +0.5D0*POK*U*W6 +0.5D0*W7
C  +0.5D0*W8)
 TEP(11)=0.5D0*(5*W11P -W1P +POK*U*W6P +0.5D0*U*W6/PK +W7P +W8P)
 TES(11)=0.5D0*(5*W11U-W1U +POK*W6 +POK*U*W6U +W7U +W8U )
 TE(12)=( 1.5D0*W12 -0.5D0*W3 -0.5D0*W14 +0.5D0*W13   )
 TEP(12)=0.5D0*( 3*W12P -W3P -W14P +W13P )
 TES(12)=0.5D0*(3*W12U -W3U -W14U +W13U )
 TE(13)=( 1.5D0*W13 -0.5D0*W3 -0.5D0*W14 +0.5D0*W12 )
 TEP(13)=0.5D0*(3*W13P -W3P -W14P +W12P)
 TES(13)=0.5D0*(3*W13U -W3U -W14U +W12U)
 TE(14)=( 1.5D0*W14 +0.5D0*W3 -0.5D0*W12 -0.5D0*W13 )
 TEP(14)=0.5D0*(3*W14P +W3P -W12P -W13P)
 TES(14)=0.5D0*(3*W14U +W3U -W12U -W13U)
 TO(1)=(1.5D0*V1 -0.5D0*V8 +0.5D0*POK*U*V11 +V12 +0.5D0*V14 )
 TOP(1)=0.5D0*(3*V1P -V8P +POK*U*V11P +0.5D0*U*V11/PK
C +2*V12P+V14P)
 TOS(1)=0.5D0*(3*V1U -V8U +POK*V11 +POK*U*V11U +2*V12U+V14U)
 TO(2)=V2
 TOP(2)=V2P
 TOS(2)=V2U
 TO(3)=(1.5D0*V3 -0.5D0*V5 -V10 -0.5D0*V11 )
 TOP(3)=0.5D0*( 3*V3P -V5P -2*V10P -V11P )
 TOS(3)=0.5D0*(3*V3U -V5U -2*V10U -V11U)
 TO(4)=(1.5D0*V4 -0.5D0*V9 -0.5D0*P2*V11 -V13 -0.5D0*PK*U*V14)
 TOP(4)=0.5D0*(3*V4P -V9P -V11 -P2*V11P-2*V13P-0.5D0*U*V14/POK
C                -PK*U*V14P )
 TOS(4)=0.5D0*(3*V4U -V9U -P2*V11U -2*V13U -PK*V14 -PK*U*V14U)
 TO(5)=(1.5D0*V5  -0.5D0*V3 +V10 +0.5D0*V11 )
 TOP(5)=0.5D0*(3*V5P -V3P +2*V10P +V11P)
 TOS(5)=0.5D0*(3*V5U -V3U +2*V10U +V11U)
 TO(6)=V6
 TOP(6)=V6P
 TOS(6)=V6U
 TO(7)=( 3.D0*V7 +0.5D0*V1 +0.5D0*P2*V2 +0.5D0*POK*U*V3
C          +0.5D0*POK*U*V5 +0.5D0*V6  +0.5D0*V8  )
 TOP(7)=0.5D0*( 6.D0*V7P +V1P +P2*V2P +V2 +0.5D0*U*V3/PK
C          +POK*U*V3P +0.5D0*U*V5/PK +POK*U*V5P +V6P  +V8P)
 TOS(7)=0.5D0*( 6.D0*V7U +V1U +P2*V2U
C   +POK*V3 +POK*U*V3U +POK*V5
C          +POK*U*V5U +V6U  +V8U )
 TO(8)=(1.5D0*V8 -0.5D0*V1 -0.5D0*POK*U*V11 -V12 -0.5D0*V14 )
 TOP(8)=0.5D0*(3*V8P-V1P -POK*U*V11P
C   -0.5D0*U*V11/PK -2*V12P -V14P)
 TOS(8)=0.5D0*(3*V8U -V1U -POK*U*V11U -POK*V11 -2*V12U -V14U)
 TO(9)=(1.5D0*V9 -0.5D0*V4+0.5D0*P2*V11 +V13 +0.5D0*PK*U*V14 )
 TOP(9)=0.5D0*(3*V9P -V4P +V11 +P2*V11P +2*V13P +PK*U*V14P
C           +0.5D0*U*V14/POK )
 TOS(9)=0.5D0*(3*V9U -V4U +P2*V11U +2*V13U +PK*U*V14U +PK*V14)
 TO(10)=(1.5D0*V10 -0.5D0*V3 +0.5D0*V5 +0.5D0*V10 +0.5D0*V11 )
 TOP(10)=0.5D0*( 3*V10P -V3P +V5P +V10P +V11P)
 TOS(10)=0.5D0*( 3*V10U -V3U +V5U +V10U +V11U)
 TO(11)=V11
```

```
      TOP(11)=V11P
      TOS(11)=V11U
      TO(12)=(1.5D0*V12 +0.5D0*V1 -0.5D0*V8 +0.5D0*V12
     C       +0.5D0*POK*U*V11 +0.5D0*V14 )
      TOP(12)=0.5D0*( 3*V12P +V1P -V8P +V12P +POK*U*V11P +0.5D0*U*
     C           V11/PK +V14P)
      TOS(12)=0.5D0*( 3*V12U +V1U -V8U +V12U
     C +POK*V11 +POK*U*V11U +V14U)
      TO(13)=(  1.5D0*V13 -0.5D0*V4 +0.5D0*V13 +0.5D0*P2*V11
     C          +0.5D0*PK*U*V14 +0.5D0*V9 )
      TOP(13)=0.5D0*( 3*V13P -V4P +V13P +V11 +P2*V11P +PK*U*V14P
     C          +0.5D0*U*V14/POK +V9P )
      TOS(13)=0.5D0*( 3*V13U -V4U +V13U +P2*V11U +PK*V14 +PK*U*V14U
     C          +V9U )
      TO(14)=V14
      TOP(14)=V14P
      TOS(14)=V14U
   50 CONTINUE
   99 CONTINUE
      IKI=0
      IF(IKI.EQ.1)WRITE (6,666) (TOP(IS),IS=1,14),(TOS(IR),IR=1,14)
  666 FORMAT (' TOP,TOS',6D15.6)
      IF(IKI.EQ.1)WRITE (6,556) (TEP(IS),IS=1,14),(TES(IR),IR=1,14)
  556 FORMAT (' TEP,TES',6D15.6)
      RETURN
      END
C THIS IS THE ELECTRON PROPAGATOR FUNCTION A(P2)
      FUNCTION XA(R)
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON/SCALE/SKL
      IF (DABS(R).LT.0.1D-6) GO TO 1
      SE=1.74517D-3
      SM=1.D0*SKL
      SM2=SM*SM
      XR=DABS(1.D0-R/SM2)
      XP=SE*(SM2-R)/R
      XA=-XR**XP
C      XA=XA*(SKL)**(2.D0*XP)
      XA=XA*SKL
      RETURN
    1 XA=1.D0
C      XA=0.0D0
      RETURN
      END
C THESE TWO FUNCTIONS DEFINE THE DERIVATIVES OF THE ELECTRON PROPAGATOR
C FUNCTION A
      FUNCTION XAP(R)
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON/SCALE/SKL
      SE=1.74517D-3
      SM=1.D0*SKL
      SM2=SM*SM
      XR=DABS(1.D0-R/SM2)
      XAP=(-SE*SM2*DLOG(XR)/R**2 -SE/R)*XA(R)
```

```
      RETURN
      END
      FUNCTION XAPP(R)
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON/SCALE/SKL
      SE=1.74517D-3
      SM=1.D0*SKL
      SM2=SM*SM
      XR=DABS(1.D0-R/SM2)
      XAPP=(2*SE*SM2*DLOG(XR)/R**3
     C +SE*SM2/((SM2-R)*R**2) +SE/R**2 )*XA(R)
     C  +(-SE*SM2*DLOG(XR)/R**2 -SE/R )*XAP(R)
      RETURN
      END
C THIS IS THE ELECTRON PROPAGATOR FUNCTION B(P2)
      FUNCTION XB(R)
      IMPLICIT REAL*8 (A-H,O-Z)
      XB=1.0D0
      RETURN
      END
      FUNCTION XBP(R)
      IMPLICIT REAL*8 (A-H,O-Z)
      XBP=0.0D0
      RETURN
      END
```

```
C THIS IS A MAIN TO FORM THE PERTURBATION SOLUTIONS
C IT DOES ALL THE DERIVATIVES OF THE INTEGRALS ANALYTICALLY.
C THIS PROGRAM WILL WORK FOR NEGATIVE MOMENTA.
C YOU MUST SUPPLY P SQUARED, U AND K SQUARED. IF P2 IS NEGATIVE
C SO MUST BE K2 AND YOU MUST GIVE THE SIGN OF THE MOMENTA
C THIS PROGRAM SENDS ITS DATA TO JC.FUND WHERE IT IS STORED
C FOR JC.MAIN TO USE.
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 K2
      COMMON/SIGNPK/SIGN
      COMMON/FUNS/XF(8,10)
      COMMON/SCALE/SKL
      WRITE(6,701)
 701  FORMAT(' P2,U,K2,SKL,SIGN')
      READ(9,*) P2,U,K2,SKL,SIGN
      PK=SIGN*DSQRT(P2*K2)
      RL=P2+2.D0*PK*U +K2
      SM1=-XA(RL)
      SM2=-XA(P2)
      WRITE(6,783) SM1,SM2,RL,P2
 783  FORMAT(' A1 AND A2',2D15.7/' P1 SQ. AND P2 SQ.',2D15.7)
      THETA=0.0
      WRITE(4,22)P2,U,K2,SKL,SIGN
  22  FORMAT(D24.16,4D11.3)
      CALL PERTF(P2,U,K2,SM1,SM2,SKL)
      CALL PERTG(P2,U,K2,SM1,SM2,SKL)
      CALL PERTH(P2,U,K2,SM1,SM2,SKL)
      DO 10 I=1,10
      XF(7,I)=-XF(7,I)
      XF(8,I)=XF(1,I)/3.D0
  10  XF(1,I)=(SM1+SM2)*XF(1,I)/2.D0
      WRITE(6,784) XF(7,1)
 784  FORMAT(' H2=',D15.7)
      DO 20 J=1,8
C      WRITE(6,33)(XF(J,I),I=1,10)
  20  WRITE(4,33)(XF(J,I),I=1,10)
  33  FORMAT(3D24.16)
      STOP
      END
C THIS PROGRAM DEFINES THE F AND I FUNCTIONS AND THEIR DERIVATIVES
      SUBROUTINE PERTF(P2,U,K2,SM1,SM2,SKL)
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 K2
      COMMON/SIGNPK/SIGN
      COMMON/FUNS/XF(8,10)
      COMMON/INT/DI(40)
C THIS IS PERTF
      EPS=2.322819D-03
      IOUT=1
      T1=1.D0
      T2=2.D0
      T3=3.D0
      T4=4.D0
      T5=5.D0
```

```
      SMA=0.5D0*(SM1+SM2)
      PK=SIGN*DSQRT(P2*K2)
      POK=DSQRT(P2/K2)
      CALL INTGRT(1,1,10,SM1,SM2,P2,U,K2)
      XF(1,1)=0.75D0*EPS*PK*DI(1)
      XF(1,2)=0.75D0*EPS*(PK*DI(2) +0.5D0*DI(1)/POK)
      XF(1,3)=0.75D0*EPS*( PK*DI(3) +DI(2)/POK
     C   -0.25D0*DI(1)/(P2*POK))
      XF(1,4)=0.75D0*EPS*(PK*DI(4) +1.5D0*DI(3)/POK
     C -0.75D0*DI(2)/(P2*POK) +T3*DI(1)/(8.D0*(P2*P2*POK)) )
      XF(1,5)=0.75D0*EPS*PK*DI(5)
      XF(1,6)=0.75D0*EPS*PK*DI(6)
      XF(1,7)=0.75D0*EPS*PK*DI(7)
      XF(1,8)=0.75D0*EPS*(PK*DI(8) +0.5D0*DI(5)/POK)
      XF(1,9)=0.75D0*EPS*(PK*DI(9) +0.5D0*DI(6)/POK)
      XF(1,10)=0.75D0*EPS*(PK*DI(10)
     C    +DI(8)/POK -0.25D0*DI(5)/(P2*POK))
      IOUT=0
      IF(IOUT.EQ.1)WRITE(6,77) (XF(1,IK),IK=1,10)
 77   FORMAT(' THE F',D24.16/3D24.16/3D24.16/3D24.16)
      RETURN
      END
C THIS PROGRAM DEFINES THE G0,G1,G2 FUNCTIONS AND DERIVATIVES
      SUBROUTINE PERTG(P2,U,K2,SM1,SM2,SKL)
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 K2
      COMMON/SIGNPK/SIGN
      COMMON/INT/DI(40)
      COMMON/FUNS/ XF(8,10)
      DATA EPS/2.322819D-03/
      AM2=SM1*SM2
      PK=SIGN*DSQRT(P2*K2)
      P4=P2*P2
      POK=DSQRT(P2/K2)
      CALL INTGRT(2,1,30,SM1,SM2,P2,U,K2)
      T1=1.0D0
      T2=2.0D0
      T3=3.0D0
      T4=4.0D0
      T5=5.0D0
      IOUT=0
      IF(IOUT.EQ.1)WRITE(6,116)P2,U
 116  FORMAT(' P2,U IN PERTG',2D18.8,I3,D14.5)
      WA=P2+T2*PK*U+K2
      WAP=T1+U/POK
      WAS=T2*PK
      WAPP=-0.5D0*U/(P2*POK)
      WAPPP=0.75D0*U/(P4*POK)
      WASP=1.D0/POK
      WASPP=-0.5D0/(P2*POK)
      W1=AM2/WA-T1
      W1P=-AM2*WAP/WA**2
      W1PP=T2*AM2*WAP*WAP/WA**3 -AM2*WAPP/WA**2
 761  FORMAT(3D15.7)
```

```
      W1PPP=-6.D0*AM2*WAP**3/WA**4 +6.D0*AM2*WAPP*WAP/WA**3
     C -AM2*WAPPP/WA**2
      W1S=-AM2*WAS/WA**2
      W1SS=T2*AM2*WAS*WAS/WA**3
      W1SSS=-6.D0*AM2*WAS**3/WA**4
      W1SP=T2*AM2*WAP*WAS/WA**3 -AM2*WASP/WA**2
      W1SSP=T2*AM2*WASP*WAS/WA**3 -6.D0*AM2*WAP*WAS**2/WA**4
     C +T2*AM2*WASP*WAS/WA**3
      W1SPP=T2*AM2*WAPP*WAS/WA**3 +T2*AM2*WAP*WASP/WA**3
     C -6.D0*AM2*WAP*WAP*WAS/WA**4 -AM2*WASPP/WA**2
     C +T2*AM2*WASP*WAP/WA**3
      W2=DLOG(DABS(AM2-WA))-DLOG(AM2)
C       W2=DLOG(DABS(1.0-WA))
      W2P=-WAP/(AM2-WA)
      W2PP=-(WAP)**2/(AM2-WA)**2
     C -WAPP/(AM2-WA)
      W2PPP=-T2*WAPP*WAP/(AM2-WA)**2
     C -T2*(WAP)**3/(AM2-WA)**3 -WAPPP/(AM2-WA)
     C -(WAPP)*(WAP)/(AM2-WA)**2
      W2S=-(WAS)/(AM2-WA)
      W2SS=-(WAS)**2/(AM2-WA)**2
      W2SSS=-T2*(WAS)**3/(AM2-WA)**3
      W2SP=-(WASP)/(AM2-WA) -(WAS)*(WAP)/(AM2-WA)**2
      W2SSP=-T2*(WASP)*(WAS)/(AM2-WA)**2
     C -T2*(WAP)*(WAS)**2/(AM2-WA)**3
      W2SPP=-T2*(WASP)*(WAP)/(AM2-WA)**2
     C -T2*(WAS)*(WAP)**2/(AM2-WA)**3
     C -(WASPP)/(AM2-WA) -(WAS)*(WAPP)/(AM2-WA)**2
      W3=AM2/P2-T1
      W3P=-AM2/P2**2
      W3PP=T2*AM2/P2**3
      W3PPP=-6.D0*AM2/P2**4
      W4=DLOG(DABS(AM2-P2))-DLOG(AM2)
C       W4=DLOG(DABS(1.D0-P2))
      W4P=-(T1)/(AM2-P2)
      W4PP=-T1/(AM2-P2)**2
      W4PPP=-T2/(AM2-P2)**3
      W=W1*W2+W3*W4
      WP=W1P*W2+W1*W2P +W3P*W4+W4P*W3
      WPP= W1PP*W2 +T2*W1P*W2P +W1*W2PP +W3PP*W4 +T2*W3P*W4P
     C +W4PP*W3
      WPPP=W1PPP*W2 +T3*W1PP*W2P +T3*W1P*W2PP +W1*W2PPP
     C +W3PPP*W4 +T3*W3PP*W4P +T3*W3P*W4PP +W3*W4PPP
      WS=W1S*W2 +W1*W2S
      WSS=W1SS*W2 +T2*W1S*W2S +W1*W2SS
      WSSS=W1SSS*W2 +T3*W1SS*W2S +T3*W1S*W2SS +W1*W2SSS
      WSP=W1SP*W2 +W1S*W2P +W1P*W2S +W1*W2SP +W1SP*W2 +W1SS*W2P
      WSSP=W1SSP*W2 +W1SS*W2P +W1SP*W2S +W1S*W2SP +W1SSP*W2
     C +W1SSS*W2P
     C +W1SP*W2S +W1S*W2SP +W1P*W2SS +W1*W2SSP +W1SP*W2S +W1SS*W2SP
      WSPP=W1SPP*W2 +W1SP*W2P +W1PP*W2S +W1P*W2SP +W1SPP*W2
     C +W1SSP*W2P
     C +W1SP*W2P +W1S*W2PP +W1P*W2SP +W1*W2SPP +W1SP*W2P +W1SS*W2PP
      CK=T2*P2*U+PK
```

```
 CKP=T2*U +1.DO/(T2*POK)
 CKPP=-0.25DO/(P2*POK)
 CKPPP=0.375DO/(P4*POK)
 CKS=T2*P2
 CKSP=T2
 XF(2,1)=1.DO +EPS*0.25DO*W +EPS*DI(1)/4.DO
 XF(2,2)=EPS*0.25DO*WP +EPS*DI(2)/4.DO
 XF(2,3)=EPS*0.25DO*WPP +EPS*DI(3)/4.DO
 XF(2,4)=EPS*0.25DO*WPPP +EPS*DI(4)/4.DO
 XF(2,5)=EPS*0.25DO*WS +EPS*DI(5)/4.DO
 XF(2,6)=EPS*0.25DO*WSS +EPS*DI(6)/4.DO
 XF(2,7)=EPS*0.25DO*WSSS +EPS*DI(7)/4.DO
 XF(2,8)=EPS*0.25DO*WSP +EPS*DI(8)/4.DO
 XF(2,9)=EPS*0.25DO*WSSP +EPS*DI(9)/4.DO
 XF(2,10)=EPS*0.25DO*WSPP +EPS*DI(10)/4.DO
 XF(3,1)=EPS*0.5DO*P2*DI(11)
 XF(3,2)=EPS*0.5DO*( DI(11)+P2*DI(12))
 XF(3,3)=EPS*0.5DO*( T2*DI(12)+P2*DI(13))
 XF(3,4)=EPS*0.5DO*( T3*DI(13)+P2*DI(14))
 XF(3,5)=EPS*0.5DO*P2*DI(15)
 XF(3,6)=EPS*0.5DO*P2*DI(16)
 XF(3,7)=EPS*0.5DO*P2*DI(17)
 XF(3,8)=EPS*0.5DO*( DI(15) +P2*DI(18))
 XF(3,9)=EPS*0.5DO*( DI(16) +DI(18) +P2*DI(19))
 XF(3,10)=EPS*0.5DO*( DI(18) +DI(18) +P2*DI(20))
 XF(4,1)=0.25DO*EPS*CK*DI(21)
C +0.25DO*EPS*PK*DI(11)
 XF(4,2)=0.25DO*EPS*( CKP*DI(21) +CK*DI(22) )
C +0.25DO*EPS*( 0.5DO*DI(11)/POK +PK*DI(12) )
 XF(4,3)=0.25DO*EPS*( CKPP*DI(21) +T2*CKP*DI(22)
C +CK*DI(23) ) +0.25DO*EPS*( -0.25DO*DI(11)/(P2*POK)
C +DI(12)/POK +PK*DI(13) )
 XF(4,4)=0.25DO*EPS*( CKPPP*DI(21) +T3*CKPP*DI(22)
C +T3*CKP*DI(23) +CK*DI(24))  +0.25DO*EPS*(
C 0.375DO*DI(11)/(P4*POK) -0.75DO*DI(12)/(P2*POK)
C +1.5DO*DI(13)/POK
C  +PK*DI(14) )
 XF(4,5)=0.25DO*EPS*( CKS*DI(21) +CK*DI(25) )
C +0.25DO*EPS*PK*DI(15)
 XF(4,6)=0.25DO*EPS*( CKS*DI(25)*T2 +CK*DI(26))
C +0.25DO*EPS*PK*DI(16)
 XF(4,7)=0.25DO*EPS*( T3*CKS*DI(26) +CK*DI(27))
C +0.25DO*EPS*PK*DI(17)
 XF(4,8)=0.25DO*EPS*( CKSP*DI(21) +CKS*DI(22)
C +CKP*DI(25) +CK*DI(28) )
C +0.25DO*EPS*( PK*DI(18) +0.5DO*DI(15)/POK)
 XF(4,9)=0.25DO*EPS*( CKSP*DI(25)*T2 +CKS*DI(28)*T2
C +CKP*DI(26) +CK*DI(29) )
C +0.25DO*EPS*( PK*DI(19) +0.5DO*DI(16)/POK )
 XF(4,10)=0.25DO*EPS*( CKSP*DI(22) +CKSP*DI(22)
C +CKS*DI(23) +CKPP*DI(25) +CKP*DI(28) +CKP*DI(28)
C +CK*DI(30) )
C +0.25DO*EPS*( 0.5DO*DI(18)/POK +PK*DI(20)
C +0.5DO*DI(18)/POK -0.25DO*DI(15)/(P2*POK) )
```

```
      RETURN
      END
C THIS PROGRAM DEFINES THE H0,H1,H2 FUNCTIONS
      SUBROUTINE PERTH(P2,U,K2,SM1,SM2,SKL)
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 K2
      COMMON/SIGNPK/SIGN
      COMMON/FUNS/XF(8,10)
      COMMON/INT/DI(40)
      EPS=2.322819D-03
      CALL INTGRT(3,1,40,SM1,SM2,P2,U,K2)
      T1=1.D0
      T2=2.D0
      T3=3.D0
      T4=4.D0
      T5=5.D0
      PK=SIGN*DSQRT(P2*K2)
      P4=P2*P2
      POK=DSQRT(P2/K2)
      SMA=(SM1+SM2)/2.D0
      CH0=-EPS*SMA*P2*K2/8.D0
      CH0P=-EPS*SMA*K2/8.D0
      CH1=-EPS*SMA*K2*(0.5D0*PK+U*P2)/8.D0
      CH1P=-EPS*SMA*K2*(0.25D0/POK+U)/8.D0
      CH1PP=-EPS*SMA*K2*(-0.125D0/(P2*POK))/8.D0
      CH1PPP=-EPS*SMA*K2*(T3/(16.D0*P4*POK))/8.D0
      CH1S=-EPS*SMA*K2*P2/8.D0
      CH1SP=-EPS*SMA*K2/8.D0
      CH1SPP=0.0D0
      BH1=-EPS*SMA*K2*PK/16.D0
      BH1P=-EPS*SMA*K2/(32.D0*POK)
      BH1PP=EPS*SMA*K2/(64.D0*P2*POK)
      BH1PPP=-1.5D0*EPS*SMA*K2/(64.D0*P4*POK)
      BH1S=0.0D0
      BH1SP=0.0D0
      BH1SPP=0.0D0
      AH2=EPS*SMA*K2/8.D0
      DH2=EPS*SMA*K2*K2/32.D0
      BH2=EPS*SMA*K2*(K2+PK*U)/16.D0
      BH2P=EPS*SMA*K2*(0.5D0*U/POK)/16.D0
      BH2PP=EPS*SMA*K2*(-0.25D0*U/(P2*POK))/16.D0
      BH2PPP=EPS*SMA*K2*(T3*U/(8.D0*P4*POK))/16.D0
      BH2S=EPS*SMA*K2*PK/16.D0
      BH2SP=EPS*SMA*K2/(32.D0*POK)
      BH2SPP=EPS*SMA*K2*(-0.5D0/(P2*POK))/32.D0
      CH2=EPS*SMA*K2*(T2*PK*U+K2)/32.D0
      CH2P=EPS*SMA*K2*U/(32.D0*POK)
      CH2PP=EPS*SMA*K2*(-U/(T2*P2*POK))/32.D0
      CH2PPP=EPS*SMA*K2*(0.75D0*U/(P4*POK))/32.D0
      CH2S=EPS*SMA*K2*(T2*PK)/32.D0
      CH2SP=EPS*SMA*K2/(32.D0*POK)
      CH2SPP=EPS*SMA*K2*(-0.5D0/(P2*POK))/32.D0
      CH2SPP=EPS*SMA*K2*(-0.5D0/(P2*POK))/32.D0
      XF(5,1)=CH0*DI(1)
```

```
      XF(5,2)=CHO*DI(2) +CHOP*DI(1)
      XF(5,3)=CHO*DI(3) +T2*CHOP*DI(2)
      XF(5,4)=CHO*DI(4) +T3*CHOP*DI(3)
      XF(5,5)=CHO*DI(5)
      XF(5,6)=CHO*DI(6)
      XF(5,7)=CHO*DI(7)
      XF(5,8)=CHO*DI(8) +CHOP*DI(5)
      XF(5,9)=CHO*DI(9) +CHOP*DI(6)
      XF(5,10)=CHO*DI(10) +T2*CHOP*DI(8)
      XF(6,1)=CH1*DI(1)+BH1*DI(21)
      XF(6,2)=CH1*DI(2) +CH1P*DI(1)
     C +BH1*DI(22) +BH1P*DI(21)
      XF(6,3)=CH1*DI(3) +T2*CH1P*DI(2) +CH1PP*DI(1)
     C   +BH1*DI(23) +T2*BH1P*DI(22) +BH1PP*DI(21)
      XF(6,4)=CH1*DI(4) +T3*CH1P*DI(3)+T3*CH1PP*DI(2)+CH1PPP*DI(1)
     C  +BH1*DI(24) +T3*BH1P*DI(23)+T3*BH1PP*DI(22)+BH1PPP*DI(21)
      XF(6,5)=CH1*DI(5)+CH1S*DI(1)+BH1*DI(25)+BH1S*DI(21)
      XF(6,6)=T2*CH1S*DI(5)+CH1*DI(6) +T2*BH1S*DI(25)+BH1*DI(26)
      XF(6,7)=T3*CH1S*DI(6)+CH1*DI(7)
     C +T3*BH1S*DI(26) +BH1*DI(27)
      XF(6,8)=CH1*DI(8) +CH1P*DI(5)+CH1SP*DI(1) +CH1S*DI(2)
     C  +BH1*DI(28) +BH1P*DI(25)+BH1SP*DI(21) +BH1S*DI(22)
      XF(6,9)=T2*CH1SP*DI(5) +CH1P*DI(6) +T2*CH1S*DI(8)+CH1*DI(9)
     C +T2*BH1SP*DI(25) +BH1P*DI(26) +T2*BH1S*DI(28) +BH1*DI(29)
      XF(6,10)=CH1PP*DI(5) +T2*CH1P*DI(8) +CH1*DI(10) +CH1SPP
     C *DI(1) +CH1SP*DI(2) +CH1SP*DI(2) +CH1S*DI(3)
     C  +BH1PP*DI(25) +T2*BH1P*DI(28) +BH1*DI(30) +BH1SPP
     C *DI(21) +BH1SP*DI(22) +BH1SP*DI(22) +BH1S*DI(23)
      XF(7,1)=CH2*DI(1)+BH2*DI(21)
      XF(7,2)=CH2*DI(2) +CH2P*DI(1)
     C +BH2*DI(22) +BH2P*DI(21)
      XF(7,3)=CH2*DI(3) +T2*CH2P*DI(2) +CH2PP*DI(1)
     C   +BH2*DI(23) +T2*BH2P*DI(22) +BH2PP*DI(21)
      XF(7,4)=CH2*DI(4) +T3*CH2P*DI(3)+T3*CH2PP*DI(2)+CH2PPP*DI(1)
     C  +BH2*DI(24) +T3*BH2P*DI(23)+T3*BH2PP*DI(22)+BH2PPP*DI(21)
      XF(7,5)=CH2*DI(5)+CH2S*DI(1)+BH2*DI(25)+BH2S*DI(21)
      XF(7,6)=T2*CH2S*DI(5)+CH2*DI(6) +T2*BH2S*DI(25)+BH2*DI(26)
      XF(7,7)=T3*CH2S*DI(6)+CH2*DI(7)
     C +T3*BH2S*DI(26) +BH2*DI(27)
      XF(7,8)=CH2*DI(8) +CH2P*DI(5)+CH2SP*DI(1) +CH2S*DI(2)
     C  +BH2*DI(28) +BH2P*DI(25)+BH2SP*DI(21) +BH2S*DI(22)
      XF(7,9)=T2*CH2SP*DI(5) +CH2P*DI(6) +T2*CH2S*DI(8)+CH2*DI(9)
     C +T2*BH2SP*DI(25) +BH2P*DI(26) +T2*BH2S*DI(28) +BH2*DI(29)
      XF(7,10)=CH2PP*DI(5) +T2*CH2P*DI(8) +CH2*DI(10) +CH2SPP
     C *DI(1) +CH2SP*DI(2) +CH2SP*DI(2) +CH2S*DI(3)
     C  +BH2PP*DI(25) +T2*BH2P*DI(28) +BH2*DI(30) +BH2SPP
     C *DI(21) +BH2SP*DI(22) +BH2SP*DI(22) +BH2S*DI(23)
      DO 151  IE=1,10
  151 XF(7,IE)=XF(7,IE)+AH2*DI(10+IE) +DH2*DI(30+IE)
      IOUT=0
      IF(IOUT.NE.1)GO TO 103
      DO 101  IF=5,7
  101 WRITE(6,102)(XF(IF,ID),ID=1,10)
 102  FORMAT(' THE H0,H1,H2',D15.7/3D15.7/3D15.7/3D15.7//)
```

```
103   CONTINUE
RETURN
END
C THE FOLLOWING PROGRAMS PERFORM INTEGRATIONS OVER BETA
      SUBROUTINE INTGRT(IZ,I,M,SM1,SM2,P2,U,K2)
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 K2
      COMMON/SIGNPK/SIGN
      COMMON/INT/DI(40)
      PK=SIGN*DSQRT(P2*K2)
      POK=DSQRT(P2/K2)
      AM2=SM1*SM2
C LOCATE SINGULARITIES
      S2=2.0D0
      S3=2.0D0
      S1=-1.D0+(AM2-P2)/(0.5D0*K2 +PK*U)
      CA=K2/4.0D0
      CB=0.5D0*K2 +PK*U
      CC=P2+PK*U +K2/4.0D0
      CE=AM2-K2/4.D0
      CF=K2/4.D0
      DE=CB*CB-4.D0*CA*CC
      IF(DE.GT.0)DESQRT=DSQRT(DE)
      IF(DE.GT.0.0D0)S2=(-CB+DSQRT(DE))/(2.0D0*CA)
      IF(DE.GT.0.0D0)S3=(-CB-DSQRT(DE))/(2.0D0*CA)
      IF(S1.LT.1.0D0.AND.S1.GT.-1.0D0) WRITE(6,43) S1
   43 FORMAT(' LAMBDA HAS A VALUE BETWEEN -1 AND 1 AT WHICH
     C (Z-1) GOES TO ZERO. LAMDA =',D15.7)
      IF(S2.LT.1.0D0.AND.S2.GT.-1.0D0)WRITE(6,44) S2
      IF(S3.LT.1.0D0.AND.S3.GT.-1.0D0) WRITE(6,44) S3
   44 FORMAT(' LAMBDA HAS A VALUE IN THE RANGE -1 TO 1 WHERE
     C Z=ZERO ; LAMBDA=',D15.7)
      BGN=-1.0D0
      WDT=2.D0
      CALL INTGRL(IZ,I,M,SM1,SM2,P2,U,K2,BGN,1.0D0)
      IOUT=0
   62 FORMAT(' THE DI INTEGRALS',3D15.7)
      IF(IOUT.EQ.2) WRITE(6,62) (DI(IR),IR=I,M)
      RETURN
      END
      SUBROUTINE  INTGRL(IZ,I,M,SM1,SM2,P2,U,K2,BGN,EN)
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 K2
      COMMON/GAUSST/XX(96),WX(96)
      COMMON/INT/DI(40)
      DIMENSION A(98)
      DO 300 N=I,M
      CALL GAUSS(BGN,EN,SLOPE)
      SGAUSS=0.0D0
      DO 100 IL=1,96
      SL=XX(IL)
      A(IL)=FN(IZ,N,SL,SM1,SM2,P2,U,K2)
  100 SGAUSS=SGAUSS +A(IL)*WX(IL)
```

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300 DI(N)=SGAUSS*SLOPE
    RETURN
    END
    SUBROUTINE GAUSS(XMIN,XMAX,SLOPE)
    IMPLICIT REAL*8 (A-H,O-Z)
    COMMON /GAUSST/ XX(96),

    DIMENSION XI(48),WI(48)
    DATA XI(1),XI(2),XI(3),XI(4),XI(5),XI(6)
   *       /.1627674484960297D-1,.4881298513604973D-1,
   *        .8129749546442556D-1,.1136958501106659D0,
   *        .1459737146548969 4D0,.1780968823676186 0D0/
    DATA XI(7),XI(8),XI(9),XI(10),XI(11),XI(12)
   *       /.2100313104605672D0,.2417431561638400 1D0,
   *        .2731988125910491 4D0,.3043649443544963 5D0,
   *        .3352085228926254 2D0,.3656968614723136 4D0/
    DATA XI(13),XI(14),XI(15),XI(16),XI(17),XI(18)
   *       /.3957976498289086 0D0,.4254789884073005 5D0,
   *        .4547094221677430 1D0,.4834579739205963 6D0,
   *        .5116941771546676 7D0,.5393881083243574 4D0/
    DATA XI(19),XI(20),XI(21),XI(22),XI(23),XI(24)
   *       /.5665104185613971 7D0,.5930323647775720 8D0,
   *        .6189258401254685 7D0,.6441634037849671 1D0,
   *        .6687183100439161 5D0,.6925645366421715 6D0/
    DATA XI(25),XI(26),XI(27),XI(28),XI(29),XI(30)
   *       /.7156768123489676 3D0,.7380306437444001 3D0,
   *        .7596023411766475 0D0,.7803690438674332 2D0,
   *        .8003087441391408 2D0,.8194003107379316 8D0/
    DATA XI(31),XI(32),XI(33),XI(34),XI(35),XI(36)
   *       /.8376235112281871 2D0,.8549590334346014 6D0,
   *        .8713885059092965 0D0,.8868945174024204 2D0,
   *        .9014606353158523 4D0,.9150714231208980 7D0/
    DATA XI(37),XI(38),XI(39),XI(40),XI(41),XI(42)
   *       /.9277124567223086 9D0,.9393703397527552 2D0,
   *        .9500327177844376 4D0,.9596882914487425 4D0,
   *        .9683268284632642 1D0,.9759391745851364 7D0/
    DATA XI(43),XI(44),XI(45),XI(46),XI(47),XI(48)
   *       /.9825172635630146 8D0,.9880541263296238 0D0,
   *        .9925439003237626 2D0,.9959818429872092 9D0,
   *        .9983643758631816 8D0,.9996895038832307 7D0/
    DATA WI(1),WI(2),WI(3),WI(4),WI(5),WI(6)
   *        /.0325506144923631 7D0,.0325161187138688 4D0,
   *         .0324471637140642 7D0,.0323438225685759 3D0,
   *         .0322062047940302 5D0,.0320344562319926 6D0/
    DATA WI(7),WI(8),WI(9),WI(10),WI(11),WI(12)
   *        /.0318287588944110 1D0,.0315893307707271 7D0,
   *         .0313164255968613 6D0,.0310103325863138 4D0,
   *         .0306713761236691 5D0,.0302999154208275 9D0/
    DATA WI(13),WI(14),WI(15),WI(16),WI(17),WI(18)
   *        /.0298963441363283 9D0,.0294610899581679 1D0,
   *         .0289946141505552 4D0,.0284974110650853 9D0,
   *         .0279700076168483 3D0,.0274129627260292 4D0/
     DATA WI(19),WI(20),WI(21),WI(22),WI(23),WI(24)
   *        /.0268268667255917 6D0,.0262123407356724 1D0,
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     *          .02557003600534936D0,.02490063322248361D0,
     *          .02420484179236469D0,.02348339908592622D0/
      DATA WI(25),WI(26),WI(27),WI(28),WI(29),WI(30)
     *         /.02273706965832937D0,.02196664443874435D0,
     *          .02117293989219130D0,.02035679715433332D0,
     *          .01951908114014502D0,.01866067962741147D0/
      DATA WI(31),WI(32),WI(33),WI(34),WI(35),WI(36)
     *         /.01778250231604526D0,.01688547986424517D0,
     *          .01597056290256229D0,.01503872102699494D0,
     *          .01409094177231486D0,.01312822956696157D0/
      DATA WI(37),WI(38),WI(39),WI(40),WI(41),WI(42)
     *         /.01215160467108832D0,.01116210209983850D0,
     *          .01016077053500842D0,.00914867123078339D0,
     *          .00812687692569876D0,.00709647079115386D0/
      DATA WI(43),WI(44),WI(45),WI(46),WI(47),WI(48)
     *         /.00605854550423596D0,.00501420274292752D0,
     *          .00396455433844469D0,.00291073181793495D0,
     *          .00185396078894692D0,.00079679206555201D0/
      SLOPE=.5D0*(XMAX-XMIN)
      YINTER=.5D0*(XMAX+XMIN)
      DO 40 I=1,96
      IF(I-48)10,10,20
   10 LL=49-I
      SGN=-1.D0
      GO TO 30
   20 LL=I-48
      SGN=1.D0
   30 X=SGN*XI(LL)
      XX(I)=SLOPE*X+YINTER
   40 WX(I)=WI(LL)
      RETURN
      END
C THIS PROGRAM DEFINES THE VARIOUS BETA INTEGRALS
      FUNCTION FN(IZ,N,S,SM1,SM2,P2,U,K2)
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8  K2,LOZ
      COMMON/SIGNPK/SIGN
      T1=1.D0
      T2=2.D0
      T3=3.D0
      T4=4.D0
      T5=5.D0
      AM2=SM1*SM2
      PK=SIGN*DSQRT(P2*K2)
      P4=P2*P2
      POK=DSQRT(P2/K2)
      YU2=AM2 -(T1-S*S)*K2/T4
      X=P2 +PK*U*(T1+S) +K2*(T1+S)*(T1+S)/T4
      Z=X/YU2
      BZ=(T1-Z)
      LOZ=DLOG(DABS(BZ))
      DP=(T1+0.5D0*U*(T1+S)/POK)/YU2
      DPP=-0.25D0*U*(T1+S)/(YU2*P2*POK)
      DPPP=T3*U*(T1+S)/(8.D0*YU2*P4*POK)
```

```
      DU=PK*(T1+S)/YU2
      DUP=0.5D0*(T1+S)/(YU2*POK)
      DUPP=-0.25D0*(T1+S)/(YU2*P2*POK)
      W1=T1/Z
      W11=-T1/Z**2
      W12=T2/Z**3
      W13=-6.D0/Z**4
      W14=24.D0/Z**5
      W15=-120.D0/Z**6
      W2=(T1/Z**2)*LOZ
      W21=-W12*LOZ +W11/BZ
      W22=-W13*LOZ +T2*W12/BZ +W11/BZ**2
      W23=-W14*LOZ +T3*W13/BZ +T3*W12/BZ**2 +T2*W11/BZ**3
      W24=-W15*LOZ +T4*W14/BZ +6.D0*W13/BZ**2 +8.D0*W12/BZ**3
     C +6.D0*W11/BZ**4
      W3=T1/(Z*BZ)
      W31=W1/BZ**2 +W11/BZ
      W32=T2*W1/BZ**3 +T2*W11/BZ**2 +W12/BZ
      W33=6.D0*W1/BZ**4+6.D0*W11/BZ**3 +T3*W12/BZ**2
     C +W13/BZ
      W4=W1*LOZ/YU2
      W41=W11*LOZ/YU2 -W1/BZ/YU2
      W42=W12*LOZ/YU2 -T2*W11/BZ/YU2 -W1/(BZ*BZ*YU2)
      W43=W13*LOZ/YU2  -T3*W12/BZ/YU2 -T3*W11/
     C (BZ*BZ*YU2) -T2*W1/(YU2*BZ**3)
      W44=W14*LOZ/YU2 -T4*W13/BZ/YU2 -6.D0*W12/(YU2*BZ**2)
     C -8.D0*W11/(YU2*BZ**3) -6.D0*W1/(YU2*BZ**4)
      W5=LOZ
      W51=-T1/BZ
      W52=-T1/BZ**2
      W53=-T2/BZ**3
      W54=-6.D0/BZ**4
      IF(IZ.NE.2) GO TO 73
      B1=(AM2-YU2)*(-W2+YU2*W4-W1)/YU2 +W5 -W4*YU2 -W4*K2/2.D0
      B2=(AM2-YU2)*(-W21+YU2*W41-W11)/YU2 +W51 -W41*YU2 -W41*
     C K2/2.D0
      B3=(AM2-YU2)*(-W22+YU2*W42-W12)/YU2 +W52 -W42*YU2 -W42*
     C K2/2.D0
      B4=(AM2-YU2)*(-W23+YU2*W43-W13)/YU2 +W53 -W43*YU2 -W43*
     C K2/2.D0
      B5=-(AM2-YU2)*(W21-YU2*W41+W11)/YU2**2
      B6=-(AM2-YU2)*(W22-YU2*W42+W12)/YU2**2
      B7=-(AM2-YU2)*(W23-YU2*W43+W13)/YU2**2
      B8=-(AM2-YU2)*(W24-YU2*W44+W14)/YU2**2
      B9=(AM2-YU2)*(W21 -YU2*W41 +W11 -W2 -W3)/YU2**2
      B10=(AM2-YU2)*(W22 -YU2*W42 +W12 -W21 -W31 )/YU2**2
      B11=(AM2-YU2)*(W23 -YU2*W43 +W13 -W22 -W32 )/YU2**2
      B12=(AM2-YU2)*(W24 -YU2*W44 +W14 -W23 -W33 )/YU2**2
      IF(N.EQ.1)FN=B1
      IF(N.EQ.2)FN=B2*DP
      IF(N.EQ.3)FN=B2*DPP +B3*DP*DP
      IF(N.EQ.4)FN=B2*DPPP +T3*B3*DPP*DP +B4*DP**3
      IF(N.EQ.5)FN=B2*DU
      IF(N.EQ.6)FN=B3*DU*DU
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      IF(N.EQ.7)FN=B4*DU**3
      IF(N.EQ.8)FN=B2*DUP +B3*DU*DP
      IF(N.EQ.9)FN=B3*DUP*DU +B3*DU*DUP +B4*DU*DU*DP
      IF(N.EQ.10)FN=B2*DUPP +T2*B3*DUP*DP +B3*DU*DPP +B4*DU*DP**2
      IF(N.EQ.11)FN=B5
      IF(N.EQ.12)FN=B6*DP
      IF(N.EQ.13)FN=B6*DPP +B7*DP**2
      IF(N.EQ.14)FN=B6*DPPP +T3*B7*DPP*DP +B8*DP**3
      IF(N.EQ.15)FN=B6*DU
      IF(N.EQ.16)FN=B7*DU**2
      IF(N.EQ.17)FN=B8*DU**3
      IF(N.EQ.18)FN=B6*DUP +B7*DU*DP
      IF(N.EQ.19)FN=T2*B7*DUP*DU +B8*DU*DU*DP
      IF(N.EQ.20)FN=B6*DUPP +T2*B7*DUP*DP +B8*DU*DP*DP
     C +B7*DU*DPP
      IF(N.EQ.21)FN=B9
      IF(N.EQ.22)FN=B10*DP
      IF(N.EQ.23)FN=B10*DPP +B11*DP**2
      IF(N.EQ.24)FN=B10*DPPP +T3*B11*DPP*DP +B12*DP**3
      IF(N.EQ.25)FN=B10*DU
      IF(N.EQ.26)FN=B11*DU**2
      IF(N.EQ.27)FN=B12*DU**3
      IF(N.EQ.28)FN=B10*DUP +B11*DU*DP
      IF(N.EQ.29)FN=T2*B11*DUP*DU +B12*DU*DU*DP
      IF(N.EQ.30)FN=B10*DUPP +T2*B11*DUP*DP +B12*DU*DP*DP
     C +B11*DU*DPP
      RETURN
   73 CONTINUE
      IF(IZ.NE.1) GO TO 72
      IF(N.EQ.1)FN=(W1+W2)/YU2
      IF(N.EQ.2)FN=(W11+W21)*DP/YU2
      IF(N.EQ.3)FN=(W11+W21)*DPP/YU2 +(W12+W22)*DP*DP/YU2
      IF(N.EQ.4)FN=(W11+W21)*DPPP/YU2 +T3*(W12+W22)*DPP*DP/YU2
     C +(W13+W23)*DP*DP*DP/YU2
      IF(N.EQ.5)FN=(W11+W21)*DU/YU2
      IF(N.EQ.6)FN=(W12+W22)*DU*DU/YU2
      IF(N.EQ.7)FN=(W13+W23)*DU*DU*DU/YU2
      IF(N.EQ.8)FN=(W11+W21)*DUP/YU2 +(W12+W22)*DP*DU/YU2
      IF(N.EQ.9)FN=(W12+W22)*DU*DUP/YU2
     C +(W13+W23)*DP*DU*DU/YU2 +(W12+W22)*DUP*DU/YU2
      IF(N.EQ.10)FN=(W11+W21)*DUPP/YU2+(W12+W22)*DPP*DU/YU2
     C +(W12+W22)*T2*DUP*DP/YU2 +(W13+W23)*DU*DP*DP/YU2
      RETURN
   72 CONTINUE
      IF(IZ.NE.3)GO TO 12
      WN1=(W21-YU2*W41+W11)/YU2**2
      WN2=(W22-YU2*W42+W12)/YU2**2
      WN3=(W23-YU2*W43+W13)/YU2**2
      WN4=(W24-YU2*W44+W14)/YU2**2
      WN5=(W2+W1)/YU2 -W4/2.D0
      WN6=(W21+W11)/YU2 -W41/2.D0
      WN7=(W22+W12)/YU2 -W42/2.D0
      WN8=(W23+W13)/YU2 -W43/2.D0
      IF(N.EQ.1)FN=WN1
```

```
      IF(N.EQ.2)FN=WN2*DP
      IF(N.EQ.3)FN=WN2*DPP +WN3*DP**2
      IF(N.EQ.4)FN=WN2*DPPP +T3*WN3*DPP*DP +WN4*DP**3
      IF(N.EQ.5)FN=WN2*DU
      IF(N.EQ.6)FN=WN3*DU*DU
      IF(N.EQ.7)FN=WN4*DU**3
      IF(N.EQ.8)FN=WN2*DUP +WN3*DU*DP
      IF(N.EQ.9)FN=T2*WN3*DUP*DU +WN4*DU*DU*DP
      IF(N.EQ.10)FN=WN2*DUPP +T2*WN3*DUP*DP +WN3*DU*DPP
     C +WN4*DP*DP*DU
      IF(N.EQ.21)FN=S*(WN1)
      IF(N.EQ.22)FN=S*(WN2*DP)
      IF(N.EQ.23)FN=S*(WN2*DPP +WN3*DP**2)
      IF(N.EQ.24)FN=S*(WN2*DPPP +T3*WN3*DPP*DP +WN4*DP**3)
      IF(N.EQ.25)FN=S*(WN2*DU)
      IF(N.EQ.26)FN=S*(WN3*DU*DU)
      IF(N.EQ.27)FN=S*(WN4*DU**3)
      IF(N.EQ.28)FN=S*(WN2*DUP +WN3*DU*DP)
      IF(N.EQ.29)FN=S*(T2*WN3*DUP*DU +WN4*DU*DU*DP)
      IF(N.EQ.30)FN=S*(WN2*DUPP +T2*WN3*DUP*DP +WN3*DU*DPP
     C +WN4*DP*DP*DU)
      IF(N.EQ.31)FN=S*S*(WN1)
      IF(N.EQ.32)FN=S*S*(WN2*DP)
      IF(N.EQ.33)FN=S*S*(WN2*DPP +WN3*DP**2)
      IF(N.EQ.34)FN=S*S*(WN2*DPPP +T3*WN3*DPP*DP +WN4*DP**3)
      IF(N.EQ.35)FN=S*S*(WN2*DU)
      IF(N.EQ.36)FN=S*S*(WN3*DU*DU)
      IF(N.EQ.37)FN=S*S*(WN4*DU**3)
      IF(N.EQ.38)FN=S*S*(WN2*DUP +WN3*DU*DP)
      IF(N.EQ.39)FN=S*S*(T2*WN3*DUP*DU +WN4*DU*DU*DP)
      IF(N.EQ.40)FN=S*S*(WN2*DUPP +T2*WN3*DUP*DP +WN3*DU*DPP
     C +WN4*DU*DP*DP)
      IF(N.EQ.11)FN=WN5
      IF(N.EQ.12)FN=WN6*DP
      IF(N.EQ.13)FN=WN6*DPP +WN7*DP**2
      IF(N.EQ.14)FN=WN6*DPPP +T3*WN7*DPP*DP +WN8*DP**3
      IF(N.EQ.15)FN=WN6*DU
      IF(N.EQ.16)FN=WN7*DU*DU
      IF(N.EQ.17)FN=WN8*DU**3
      IF(N.EQ.18)FN=WN6*DUP +WN7*DU*DP
      IF(N.EQ.19)FN=T2*WN7*DUP*DU +WN8*DU*DU*DP
      IF(N.EQ.20)FN=WN6*DUPP +T2*WN7*DUP*DP +WN7*DU*DPP
     C +WN8*DU*DP*DP
      RETURN
   12 CONTINUE
      WRITE(6,87)
   87 FORMAT(' CORRECT VALUE IF IZ IN FNNS NOT FOUND')
      RETURN
      END
C THIS DEFINES THE ELECTRON FUNCTION A
      FUNCTION XA(R)
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON/SCALE/SKL
      IF (DABS(R).LT.0.1D-6) GO TO 1
```

```
      SE=1.74517D-3
      SM=1.D0*SKL
      SM2=SM*SM
      XR=DABS(1.D0-R/SM2)
      XP=SE*(SM2-R)/R
      XA=-XR**XP
C      XA=XA*(SKL)**(2.D0*XP)
      XA=XA*SKL
      RETURN
    1 XA=1.D0
C      XA=0.0D0
      RETURN
      END
```

APPENDIX D

CALCULATION OF FOUR DIMENSIONAL INTEGRALS

In the electron self energy integral, $\Sigma(\bar{p})$, and the vacuum polarization integral, $\Pi^{\mu\nu}(p^2)$, various types of four dimensional integrals occur. These integrals may be evaluated if a Wick rotation is made to cause the real axis of the fourth component of the momentum to lie along the imaginary axis. This can be done by substituting

$$p^o \to ip^4.$$

The integral can be performed than over four dimensional Euclidean space. Thus,

$$d^4p = dp^o dp^1 dp^2 dp^3 \to idp^4 dp^1 dp^2 dp^3.$$

The momenta variables can be expressed as four dimensional hyperspherical coordinates so that,

$$\begin{aligned} p^1 &= r^{\frac{1}{2}} \sin\theta \cos\phi \sin\chi \\ p^2 &= r^{\frac{1}{2}} \sin\theta \sin\phi \sin\chi \\ p^3 &= r^{\frac{1}{2}} \cos\theta \sin\chi \\ p^4 &= r^{\frac{1}{2}} \cos\chi \end{aligned} \tag{D-1}$$

where $r = -p^2$. (Hence r is a positive quantity.) The Jacobian of this transformation from Cartesian to hyperspherical coordinates is given by

$$J = r^{3/2} \sin\theta \sin^2\chi$$

The ranges of integration are,

$$0 \leqslant \phi \leqslant 2\pi$$

$$0 \leqslant \theta \leqslant \pi$$

$$0 \leqslant \chi \leqslant \pi$$

$$0 \leqslant r \leqslant \infty \ .$$

In summary, we can evaluate the following integral by an integration over the hyperspherical coordinates,

$$\int f(p^2)d^4p = \frac{i}{2} \int f(r)r^{3/2} \sin\theta \sin^2\chi \ dr \ d\theta \ d\phi \ dp. \qquad \text{(D-2)}$$

A further complication occurs when a second independent vector, $\bar{k}$, is introduced. Let $\bar{k}$ lie along the p^4 axis, then the angle variable relating $\bar{p}$ do $\bar{k}$ is

$$u = \frac{p \cdot k}{(p^2k^2)^{\frac{1}{2}}} = \cos\chi \ . \qquad \text{(D-3)}$$

We would like to consider the evaluation of integrals which involve functions of this angle variable, as in

$$\int f(p^2, \ u, \ k^2)d^4p.$$

Considerable simplification of these integrals is obtained by expanding the integrands in terms of Gegenbauer polynomials. These have the following orthogonality relation,

$$\int_{-1}^{1} C_a^1(u) \ C_b^1(u) \ (1-u^2)^{\frac{1}{2}} \ du = \frac{\Pi}{2} \ \delta_{ab}. \qquad \text{(D-4)}$$

The first three Gegenbauer polynomials are,

$$C_o^1 = 1$$

$$C_1^1 = 2u$$

$$C_2^1 = - 1 + 4u^2.$$

The higher order polynomials can be generated from the recursion relation,

$$2(1-n)u\ C_n^1\ (u) = (2+n-1)\ C_{n-1}^1 + (n+1)\ C_{n+1}^1. \tag{D-5}$$

The employment of the Gegenbauer polynomials permits the evaluation of such integrals as

$$I^{\mu\nu} = \int f(p^2)p^\mu p^\nu d^4p$$

$$= \frac{1}{2}\int f(r)p^\mu p^\nu\ r\ dr(1-u^2)^{\frac{1}{2}}\ du\ \sin\theta\ d\theta d\phi.$$

This is best evaluated by components. There are sixteen possible components. Consider first,

$$I^{11} = \frac{1}{2}\int f(r)p^1p^1 rdr(1-u^2)^{\frac{1}{2}}\ du\ \sin\ \theta\ d\theta d\phi$$

$$= \frac{1}{2}\int f(r)r^2dr(1-u^2)(1-u^2)^{\frac{1}{2}}du\ \sin^3\theta\ d\theta\ \cos^2\phi d\phi$$

$$= \frac{2\pi i}{3}\int f(r)r^2dr(1-u^2)(1-u^2)du$$

$$= \frac{2\pi i}{3}\int f(r)r^2dr(C_o^{12} - \frac{1}{4}\ C_1^{12})(1-u^2)^{\frac{1}{2}}du$$

$$= \frac{\pi^2 i}{4}\int f(r)r^2dr.$$

Similar steps taken to evaluate the other components lead to

$$I^{11} = I^{22} = I^{33} = I^{44}$$

and also

$$I^{\alpha\beta} = 0 \text{ when } \alpha \neq \beta.$$

It follows that

$$I^{\mu\nu} = g^{\mu\nu} \frac{\pi^2 i}{4} \int f(4)^2 dr.$$

It is no more difficult to evaluate a more general situation where f is a function of u as well as r. In this case the expression of the integrans in terms of the Gegenbauer polynomials would be altered but the integration over u could be performed with the same ease due to the orthogonality relations.

There is one last useful maneuver that should be mentioned. In both the $\Pi^{\mu\nu}(k^2)$ and $\Sigma(\bar{p})$ integrals the expression $(\bar{p}-\bar{k})^2$ occurs in the denominators. This expression has an expansion in terms of Gegenbauer polynomials. Let

$$(\bar{p}-\bar{k})^{-2} = -\frac{1}{r_>}(1 - 2uz + z^2)$$

where

$$z = \left(\frac{r_<}{r_>}\right)^{\frac{1}{2}}$$

$$r_< = \text{the lesser of } p^p, k^2$$

$$r_> = \text{the greater of } p^2, k^2.$$

Since $(1 - 2\mu z + z^2)^{-1} = \sum_{n=0}^{\infty} C_n^1(u)z^n$, we find,

$$(\bar{p}-\bar{k})^{-2} = -\frac{1}{r_>}\sum_{n=0}^{\infty} C_n^1(u)z^n.$$

This enables one to evaluate such integrals as,

$$\int \frac{f(p^2,u,k^2)d^4p}{(\bar{p}-\bar{k})} = -\frac{i}{2}\int f(r,u,k^2)\sum_{n=0}^{\infty} C_n^1 \frac{z^n}{r_>} \; rdr(1-u^2)^{\frac{1}{2}}du \sin\theta \, d\theta d\phi.$$

For the purposes of illustration, consider the case where $f = f(p^2)$,

$$\int \frac{f(p^2)}{(\bar{p}-\bar{k})^2} \; d^4p = -\frac{i}{2}\int f(r) \sum_{n=0}^{\infty} C_n^1 \frac{z^n}{r_>} \; r \, dr(1-u^2)^{\frac{1}{2}}du \sin\theta \, d\theta d\phi$$

$$= -\Pi i \int f(r) \; C_1^1 \sum_{n=0}^{\infty} C_n^1 \frac{z^n}{r_>} \; r \, dr(1-u^2)^{\frac{1}{2}}du$$

$$= -\frac{\Pi^2 i}{2}\int f(r)\,\frac{r}{r_>}\,dr.$$

These methods described here are sufficient to calculate any of the four dimensional integrals which appear in $\Sigma(\bar{p})$ and $\Gamma^{\mu\nu}(k^2)$, provided that the form of the function $f(p^2,u,k^2)$ is known. A tabulation of the integrals needed to calculate the self energy of the electron, $\Sigma(\bar{p})$, appears in the work of Green et al.[28] A tabulation of the integrals needed to calculate the vacuum polarization, $\Gamma^{\mu\nu}(k^2)$, is provided in the work of Yock.[15]

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Joan F. Cartier was born in Toronto, Canada, in 1950 but was for the most part raised in the out-islands of the Bahamas. She obtained a Bachelor of Science degree with High Honors from the University of Florida in 1975 while studying chemistry.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosopny.

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April 1983

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